Locally Convex Analysis in Differential Geometry

Minicourse: Who is afraid of Infinite Dimensions?

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October 2025 – Lyon

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Instruction Manual

These lecture notes are based on the textbooks [7,8,10–14] as well as the encyclopaedic [6]. The lecture notes [15,17,18] have also been instrumental. Moreover, they harbour many scattered ideas and, undoubtedly, numerous typographical (and by all likelihood also some mathematical) errors introduced by the author. Moreover, many interesting and sometimes crucial facts have been left as exercises. On a first reading, most of them can be assumed as blackboxes or ignored safely, but they are of course crucial for a deeper understanding. The unsure or otherwise confused reader should thus not be afraid to reach out to discuss. The main goal of these notes is to provide a gentle introduction to locally convex analysis for the working differential geometer, while also ultimately covering some advanced topics regarding tensor products and the arcane notion of nuclearity. As such, we assume a certain familiarity with concepts from point set topology, but stay on the lighter side regarding functional analysis. Finally, the author's background results in a general philosophy on *reducing problems to inequalities*, whereas the puristic topological point of view takes a secondary role.

If you are still here: Why are you reading the instruction manual? What kind of dodgy lecture notes require a manual anyway? Who sold you these? Be careful out there. There are *analysts* about. One anyway.

1 Beyond Normed Spaces

Throughout the text, the principal assumption is that we are working with vector spaces over the field $\mathbb{C}$ of complex numbers. Accordingly, linear mappings are complex linear and most mappings take values in the complex numbers. The real theory is largely analogous, and we indicate whenever it is not.

While normed spaces provide a fruitful and vast framework, they turn out to be insufficient to capture a number of natural phenomenona within Differential Geometry and Analysis. In this preliminary section, we explore several such examples.

Example 1.1 Let $K \subseteq \mathbb{C}^n$ be compact. Then we may endow the space of complex-valued continuous functions

$$\mathscr{C}(K) := \big\{ f \colon K \longrightarrow \mathbb{C} \;\big|\; f \text{ is continuous} \big\}$$

with the supremum norm

$$\|f\|_K := \max_{z \in K} \big| f(z) \big| \tag{1.1}$$

to obtain a Banach space. The corresponding topology of *uniform convergence* may be defined by taking the collection of open balls

$$\mathrm{B}_r(f) := \big\{ g \in \mathscr{C}(K) \colon \|f - g\|_K < r \big\}$$

with $r > 0$ as a topological basis. That is to say, a subset $\mathcal{U} \subseteq \mathscr{C}(K)$ is open iff it may be written as a union of open balls. The compactness of K is used to ensure that the maximum within (1.1) is well defined. Hence, it is a natural question how one should go about topologizing

$$\mathscr{C}(U) := \big\{ f \colon U \longrightarrow \mathbb{C} \;\big|\; f \text{ is continuous} \big\}$$

for, say open, subsets $U \subseteq \mathbb{C}^n$. Two natural wishes are the continuity of the restriction mappings

$$\cdot \Big|_K \colon \mathscr{C}(U) \longrightarrow \mathscr{C}(K) \tag{1.2}$$

for any compact set $K \subseteq U$ and the completeness of the resulting space. Taking a step back, we remember that continuity is a local property, and as such not only preserved by uniform convergence, but also by *locally uniform convergence*. By virtue of local compactness of $\mathbb{C}^d$, this may be rephrased as uniform convergence on compact subsets. That is to say, a net within $\mathscr{C}(U)$ should converge iff all of its images under the restriction mappings (1.2) are convergent in the normed spaces $\mathscr{C}(K)$. By choosing an exhaustion $(K_n)_n$ of U by compact subsets, this may in turn be formalized by introducing the metric

$$d(f, g) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}} \qquad \text{for all } f, g \in \mathscr{C}(U). \tag{1.3}$$

The resulting metric is invariant under translations and turns $\mathscr{C}(U)$ into a complete metric space by Exercise 1.3. We have moreover built in the continuity of (1.2) into the definition. However, from a topological point of view, the open sets seem at first glance rather complicated. Taking a closer look, we realize that the "open cylinders"

$$\mathrm{B}_{K,r}(f) := \big\{ g \in \mathscr{C}(U) \colon \|f - g\|_K < r \big\} \subseteq \mathscr{C}(U) \tag{1.4}$$

are actually open for all compact subsets $K \subseteq U$, $r > 0$ and $f \in \mathscr{C}(U)$. Indeed, if $g \in \mathrm{B}_{K_n,r}(f)$ and $h \in \mathrm{B}_{2^{-n}\cdot r_0}(g)$ is in the metric ball with radius $r_0 < 1$ around g, then using that

$$\phi\colon [0,\infty) \longrightarrow [0,1), \quad \phi(x) := \frac{x}{1+x}$$

is strictly increasing with inverse $\phi^{-1}(y) = y/(1-y)$ yields

$$\frac{\|g-h\|_{K_n}}{1+\|g-h\|_{K_n}} < r_0 \qquad \Longrightarrow \qquad \|g-h\|_{K_n} < \frac{r_0}{1-r_0} = \phi^{-1}(r_0)$$

and thus

$$\|f-h\|_{K_n} \leq \|f-g\|_{K_n} + \|g-h\|_{K_n} < \|f-g\|_{K_n} + \phi^{-1}(r_0)$$

Hence, setting

$$r_0 := \phi\big(r - \|f-g\|_{K_n}\big) \in [0,1)$$

produces a metric ball contained within $\mathrm{B}_{K,r}(f)$, proving the openness. Conversely, every $g \in \mathrm{B}_r(f)$ is contained within $\mathrm{B}_{K,\phi^{-1}(2^{-n}r)}(f)$ for all $n \in \mathbb{N}_0$ and thus we may forget about the metric balls in the sequel and only work with the open cylinders. Having established this, we get another pleasant property of this topology essentially for free. Namely, the continuity of the pointwise vector space operations

$$+\colon \mathscr{C}(U) \times \mathscr{C}(U) \longrightarrow \mathscr{C}(U)$$
$$\cdot\colon \mathbb{C} \times \mathscr{C}(U) \longrightarrow \mathscr{C}(U)$$

and even of the pointwise multiplication. Geometrically, the sets (1.4) may indeed be thought of as cylinders, as it is absolutely convex and $\|\cdot\|_K$ comes with the typically sizable kernel

$$\ker\big(\|\cdot\|_K\big) = \big\{f \in \mathscr{C}(U)\colon f\big|_K \equiv 0\big\}.$$

Algebraically, the mappings $\|\cdot\|_K$ thus constitute seminorms on $\mathscr{C}(U)$. That is to say, they fulfil all properties of a norm, except for having trivial kernel. By what we have shown, they may be used as a basis for the metric topology. This correspondence between systems of seminorms and topologies is at the heart of locally convex topologies and we shall explore this in Section 2.

It is instructive to work out the details of the arguments used in the prior discussion.

Exercise 1.2 Let X be a metric space. Prove that a subset $U \subseteq X$ is open iff it is a union of open balls.

Exercise 1.3 Let X be a locally compact topological space and $f, f_\alpha\colon X \longrightarrow \mathbb{C}$ be mappings for all $\alpha \in J$ for some directed set J. Prove that the following are equivalent:

i.) For every compact set $K \subseteq X$, the net of the restrictions $(f_\alpha\big|_K)$ converges uniformly to the function $f\big|_K$.

ii.) Every $p \in X$ has an open neighbourhood $U \subseteq X$ such that the net of restrictions $(f_\alpha\big|_U)$ converges uniformly to the function $f\big|_U$.

Exercise 1.4 Let $U \subseteq \mathbb{R}^d$ or $U \subseteq \mathbb{C}^n$ be open. Prove that the metric (1.3) turns $\mathscr{C}(U)$ into a complete metric space.

Before turning to the general theory, we discuss some more remarkable observations that necessitate the investigation of functional analysis beyond norms.

Example 1.5 Consider the space $\mathscr{C}^\infty(\mathbb{R})$ of smooth functions on the real line and the differentiation operator

$$\mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}x} : \mathscr{C}^\infty(\mathbb{R}) \longrightarrow \mathscr{C}^\infty(\mathbb{R}).$$

Then the exponential functions

$$f_\alpha : \mathbb{R} \longrightarrow \mathbb{C}, \quad f_\alpha(x) := \exp\bigl(\alpha x\bigr)$$

constitute eigenfunctions of D for all $\alpha \in \mathbb{C}$. Hence, even if we somehow succeed in endowing $\mathscr{C}^\infty(\mathbb{R})$ with the structure of a Banach algebra, then D is necessarily discontinuous by virtue of its unbounded spectrum.

Another strange feature of normed spaces is that some are never complete, regardless of how one chooses the norm.

Lemma 1.6 *Let V be a normed vector space of countably infinite dimension. Then V is incomplete.*

PROOF: Assuming otherwise, we consider the family of closed subspaces

$$V_n := \operatorname{span}\{e_1, \ldots, e_n\} \qquad \text{for } n \in \mathbb{N}_0$$

and where $(e_n)_n \subseteq V$ is any basis. Then

$$V = \bigcup_{n=0}^{\infty} V_n,$$

and thus there exists some $n_0 \in \mathbb{N}$ such that $V_{n_0}^\circ \neq \emptyset$ by Baire's Theorem. This is a contradiction, as $ze_{n_0+1} \notin V_{n_0}$ for all $z \in \mathbb{C} \setminus \{0\}$. □

Finally, we recast some classical complex analysis in a locally convex light.

Exercise 1.7 Let $U \subseteq \mathbb{C}^d$ be open. Prove that the space of holomorphic functions $\mathscr{H}(U)$ constitutes a closed subspace of $\mathscr{C}(U)$.

Recall Montel's Theorem.

Theorem 1.8 (Montel) *Let $U \subseteq \mathbb{C}^d$ be a domain. Then every locally bounded sequence $(f_n)_n \subseteq \mathscr{H}(U)$ has a convergent subsequence.*

Classically, one speaks of normality of such families of functions. The reason for this is that Montel's work actually predates the abstract notion of compactness. Either way, we get the following consequence.

Lemma 1.9 *Let $U \subseteq \mathbb{C}^d$ be a domain. Then the topology of locally uniform convergence on the space of holomorphic functions $\mathscr{H}(U)$ on U may not be induced by a norm.*

PROOF: Assuming otherwise, we get that the open unit ball

$$\mathrm{B}_1(0) \subseteq \mathscr{H}(U)$$

constitutes a bounded subset of $\mathscr{H}(U)$. By Montel's Theorem, this implies the compactness of its closure $\mathrm{B}_1(0)^{\mathrm{cl}}$, which is only possible if $\mathscr{H}(U)$ is finite-dimensional. This is a contradiction to the holomorphicity of the monomials. □

In this sense, the space $\mathscr{H}(U)$ behaves much like a finite-dimensional space. The abstract underlying reason is its nuclearity, which is the content of Section 4.

2 From Seminorms to Locally Convex Spaces

Having convinced ourselves of the wonders and richness of the locally convex world within some examples, we now venture into the its abstract horrors. Most of the material is standard and can be found in all of the textbooks mentioned in the introduction. That being said, a particularly conceptually clean presentation is [11, Ch. 3]. As already noted, we shall place our focus on seminorms, as they ultimately frame the abstract results in a way convenient for the investigation of concrete problems.

Definition 2.1 (Seminorms) *Let V be a vector space. A mapping $\mathrm{q}\colon V \longrightarrow [0, \infty)$ is called a seminorm if:*

i.) It is absolutely 1-homogeneous, i.e.

$$\mathrm{q}\big(\lambda \cdot v\big) = |\lambda| \cdot \mathrm{q}(v) \qquad \textit{for all } v \in V, \lambda \in \mathbb{C}. \tag{2.1}$$

ii.) It fulfils the triangle inequality

$$\mathrm{q}(v + w) \leq \mathrm{q}(v) + \mathrm{q}(w) \qquad \textit{for all } v, w \in V.$$

Note that (2.1) implies $\mathrm{q}(0) = 0$, as we are working in characteristic different from two. Associated to any seminorm q, we define its collection of open cylinders

$$\mathrm{B}_{\mathrm{q},r}(v) := \big\{w \in V : \mathrm{q}(v - w) < r\big\}$$

with radius $r > 0$. We proceed with a list of examples.

Example 2.2 (Seminorms)

i.) Every norm is a seminorm. The corresponding open cylinders are the open balls we are all used to.

ii.) Let $v'\colon V \longrightarrow \mathbb{C}$ be linear. Then

$$|v'| := |\cdot| \circ v'$$

is a seminorm of particularly simple type. An instructive example is the projection

$$v'\colon \mathbb{R}^2 \longrightarrow \mathbb{C}, \quad v'(x^1, x^2) := x^1$$

onto the first component. Then

$$\ker|v'| = \ker v' = \{0\} \times \mathbb{R}$$

as well as

$$|v'|\big((x^1, x^2) - (y^1, y^2)\big) = \big|x^1 - y^1\big| \qquad \text{for all } x, y \in \mathbb{R}^2.$$

Consequently,

$$\mathrm{B}_{|v'|,r}(x) = \big\{y \in \mathbb{R}^2 \colon |x^1 - y^1| < r\big\} = (x^1 - r, x^1 + r) \times \mathbb{R}$$

really constitutes a cylinder in this case.

iii.) Let $U \subseteq \mathbb{C}^d$ be open. Then mappings $\|\cdot\|_K$ constitute seminorms on $\mathscr{C}(U)$ for all compact subsets $K \subseteq U$. More generally, we get seminorms on $\mathscr{H}(U)$ by setting

$$\|f\|_{K,\alpha} := \max_{|\beta|\leq|\alpha|} \big\|\partial^\alpha f\big\|_K$$

for any multi-index $\alpha \in \mathbb{N}_0^d$. Replacing complex derivatives with real ones, the same formula also defines seminorms on $\mathscr{C}^\infty(U)$. By virtue of the identity principle, if U is connected, then $\|\cdot\|_K$ is even a norm on $\mathscr{H}(U)$ for every compact $K \subseteq U$ with non-empty open interior.

iv.) For every $r > 0$, the mappings

$$\mathrm{q}_r\left(\sum_{n=0}^{\infty} a_n \cdot z^n\right) := \sum_{n=0}^{\infty} |a_n| \cdot r^n$$

define norms on $\mathscr{H}(\mathbb{C})$.

v.) Let $n, m \in \mathbb{N}_0$ and define

$$\mathrm{r}_{n,m}(f) := \sup_{x\in\mathbb{R}} (1 + x^2)^m \cdot \big|f^n(x)\big|$$

for $f \in \mathscr{C}^\infty(\mathbb{R})$. Then $\mathrm{r}_{n,m}(f) < \infty$ for all $n, m \in \mathbb{N}_0$ is equivalent to f being an element of the classical Schwartz space. Once again, we are actually dealing with norms instead of just seminorms.

It is a common theme within locally convex analysis to rephrase the membership in some function space as a seminorm condition. As we shall see, this then gives rise to an associated locally convex topology, which often turns out to have desirable properties such as completeness.

Exercise 2.3 Show that the systems of seminorms

$$\big\{\|\cdot\|_K \colon K \subseteq \mathbb{C} \text{ compact}\big\}$$

and

$$\big\{\mathrm{q}_r \colon r > 0\big\}$$

are equivalent on $\mathscr{H}(\mathbb{C})$. That is to say, for every compact subset $K \subseteq \mathbb{C}$ there exists some $r > 0$ with

$$\|f\|_K \leq \mathrm{q}_r(f) \qquad \text{for all } f \in \mathscr{H}(\mathbb{C})$$

and, conversely, for every $r > 0$ there exists a compact set $K \subseteq \mathbb{C}^d$ such that

$$\mathrm{q}_r(f) \leq \|f\|_K \qquad \text{for all } f \in \mathscr{H}(\mathbb{C}).$$

Remarkably, one may reconstruct a seminorm from its open unit cylinder. This correspondence between algebraic gadgets and geometric objects is established by means of the Minkowski functional.

Definition 2.4 (Minkowski functional) *Let V be a vector space and $U \subseteq X$ be absorbing, i.e.*

$$\bigcup_{r>0} rU = \bigcup_{r>0} \big\{ r \cdot v \colon v \in U \big\} = V.$$

Then

$$\mathrm{p}_U \colon V \longrightarrow [0, \infty), \quad \mathrm{p}_U(v) := \inf \big\{ r > 0 \colon v \in r \cdot U \big\} \tag{2.2}$$

is called the Minkowski functional of U.

Recall that a subset C of a vector space V is called convex if the line segment

$$[v, w] := \big\{ \lambda \cdot v + (1 - \lambda) w \colon \lambda \in [0, 1] \big\} \subseteq C$$

for all $v, w \in C$. It is called balanced (or circled) if

$$\eta \cdot U = U \qquad \text{for all } \eta \in \mathbb{C}, |\eta| = 1.$$

Combining both notions, one speaks of absolute convexity of U.

Proposition 2.5 *Let V be a vector space and $U \subseteq V$ be absorbing.*

i.) The Minkowski functional (2.2) *is well-defined.*

ii.) If U is convex, then p_U is sublinear, i.e.

$$\mathrm{p}_U(v + w) \leq \mathrm{p}_U(v) + \mathrm{p}_U(w) \quad \textit{and} \quad \mathrm{p}(r \cdot v) = r \cdot \mathrm{p}(v)$$

for all $v, w \in V$ and $r \geq 0$.

iii.) If U is convex and balanced, then p_U is a seminorm. Moreover,

$$\mathrm{B}_{\mathrm{p}_U, 1}(0) \subseteq U \subseteq \mathrm{B}_{\mathrm{p}_U, 1}(0)^{\mathrm{cl}}.$$

iv.) If q is a seminorm, then the open cylinders $\mathrm{B}_{\mathrm{p}, r}(0)$ are absorbing, convex and balanced for all $r > 0$. Moreover,

$$\mathrm{p}_{\mathrm{B}_{\mathrm{q},1}(0)} = \mathrm{q} = \mathrm{p}_{\mathrm{B}_{\mathrm{q},1}(0)^{\mathrm{cl}}}, \tag{2.3}$$

where we define

$$\mathrm{B}_{\mathrm{q},1}(0)^{\mathrm{cl}} = \big\{ v \in V \colon \mathrm{q} \leq 1 \big\}.$$

Exercise 2.6 Prove Proposition 2.5.

The central idea behind locally convex topologies now to use the open cylinders corresponding to a collection of seminorms as the basis of a topology. This is also where the phrase *locally convex* stems from: The origin possesses a neighbourhood basis consisting of absolutely convex sets! By Proposition 2.5, this demand automatically leads towards seminorms.

Returning to this idea, we define a topology on V associated to some collection $\mathcal{P}$ of seminorms by first taking finite intersections of open cylinders and then arbitrary unions of the resulting sets. There is a simple condition on the level of seminorms, as to when we may skip the first step, i.e. are actually dealing with a basis of the topology. Note that any set of seminorms is, in particular, a set of real-valued mappings defined on a joint domain. As such, it carries the pointwisely defined partial order, i.e.

$$\mathrm{q} \leq \mathrm{p} \qquad \Longleftrightarrow \qquad \mathrm{q}(v) \leq \mathrm{p}(v) \quad \text{for all } v \in V.$$

Recall that a partially ordered set $(\mathcal{P}, \leq)$ is called a directed if , i.e. for all $\mathrm{q}_1, \mathrm{q}_2 \in \mathcal{P}$ there exists $\mathrm{p} \in \mathcal{P}$ with $\mathrm{q}_1 \leq \mathrm{p}$ and $\mathrm{q}_2 \leq \mathrm{p}$.

Lemma 2.7 *Let V be a vector space and $\mathcal{P}$ a set of seminorms on V such that $(\mathcal{P}, \leq)$ is a directed set. Then every finite intersection of elements within the set*

$$\mathcal{B} := \big\{ \mathrm{B}_{\mathrm{q},r}(v) \colon \mathrm{q} \in \mathcal{P}, r > 0, v \in V \big\} \tag{2.4}$$

may be written as a union of elements of $\mathcal{B}$.

PROOF: It suffices to prove the claim for intersections of two open cylinders. To this end, let $\mathrm{q}_1, \mathrm{q}_2 \in \mathcal{P}$, $r_1, r_2 > 0$ and $v_1, v_2 \in V$ such that

$$U := \mathrm{B}_{\mathrm{q}_1,r_1}(v_1) \cap \mathrm{B}_{\mathrm{q}_2,r_2}(v_2) \neq \emptyset.$$

By assumption, we find a joint upper bound $\mathrm{p} \in \mathcal{P}$ of q_1 and q_2. Let $v \in U$ and

$$r_v := \min\big\{ r_1 - \mathrm{q}_1(v - v_1), r_2 - \mathrm{q}_2(v - v_2) \big\} > 0.$$

Then $\mathrm{B}_{\mathrm{p},r_v}(v) \subseteq U$, as

$$\mathrm{q}_1\big(v - w\big) \leq \mathrm{q}_1\big(v - v_1\big) + \mathrm{p}\big(v_1 - w\big) < \mathrm{q}_1\big(v - v_1\big) + r_v < r_1$$

and, analogously, $\mathrm{q}_2(v - w) < r_2$ for all $w \in \mathrm{B}_{\mathrm{p},r}(v)$. Consequently, we get

$$U = \bigcup_{v \in U} \mathrm{B}_{\mathrm{p},r_v}(v),$$

as $v \in \mathrm{B}_{\mathrm{p},r_v}(v)$ for all $v \in U$. $\square$

It is customary to call a collection of seminorms $\mathcal{P}$ *filtrating* if $(\mathcal{P}, \leq)$ is directed. As linear combinations of seminorms with non-negative coefficients are again seminorms, the collection of all seminorms is always filtrating. We are now in a position to define locally convex spaces.

Definition 2.8 (Locally convex space) *A vector space V endowed with a topology is called locally convex if there exists a filtrating system of seminorms $\mathcal{P}$ on V such that*

$$\mathcal{B} := \big\{ \mathrm{B}_{\mathrm{q},r}(v) \colon \mathrm{q} \in \mathcal{P}, r > 0, v \in V \big\}$$

constitutes a basis of the topology. We then call $\mathcal{P}$ a defining system of seminorms for V. Moreover, we write

$$\mathrm{cs}(V) := \big\{ \mathrm{q} \colon V \longrightarrow \mathbb{R} \,\big|\, \mathrm{q} \textit{ is a continuous seminorm} \big\}$$

for the set of all continuous seminorms of a locally convex space.

This definition raises some questions, the answers of which we collect before proceeding.

Proposition 2.9 *Let V be locally convex with defining system of seminorms $\mathcal{P}$.*

i.) The locally convex space V is a topological vector space, i.e.

$$+\colon V \times V \longrightarrow V \qquad \text{and} \qquad \cdot\colon \mathbb{C} \times V \longrightarrow V$$

are continuous, where we endow $V \times V$ and $\mathbb{C} \times V$ with the product topology.

ii.) For every $v \in V$, the translation

$$\tau_v\colon V \longrightarrow V, \quad \tau_v(w) := v + w$$

constitutes a linear homeomorphism.

iii.) A seminorm q on V is continuous iff there exist $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ and $c_1, \ldots, c_n > 0$ such that

$$\mathrm{q} \leq c_1 \cdot \mathrm{p}_1 + \cdots + c_n \cdot \mathrm{p}_n. \tag{2.5}$$

iv.) The locally convex topology induced by the collection of all continuous seminorm coincides with the topology of V.

v.) A subset $U \subseteq V$ is open iff for every $v \in U$ there exists $\mathrm{q} \in \mathrm{cs}(V)$ and some $r > 0$ with

$$\mathrm{B}_{\mathrm{q},r}(v) \subseteq U.$$

vi.) The locally convex space V is Hausdorff iff for every $v \in V$ there exists $\mathrm{q} \in \mathrm{cs}(V)$ with $\mathrm{q}(v) > 0$.

Proof: Let $\mathrm{q} \in \mathcal{P}$, $r > 0$ and $v \in V$. We have to prove that

$$+^{-1}\big(\mathrm{B}_{\mathrm{q},r}(v)\big) = \big\{(v_1, v_2) \in V \times V \colon \mathrm{q}(v_1 + v_2 - v) < r\big\}$$

is open. Let $(v_1, v_2) \in +^{-1}(\mathrm{B}_{\mathrm{q},r}(v))$ and set $\delta := \mathrm{q}(v_1 + v_2 - v)$. We claim

$$B := \mathrm{B}_{\mathrm{q},(r-\delta)/2}(v_1) \times \mathrm{B}_{\mathrm{q},(r-\delta)/2}(v_2) \subseteq +^{-1}\big(\mathrm{B}_{\mathrm{q},r}(v)\big).$$

Indeed, if $(w_1, w_2) \in B$, then by the triangle inequality

$$\mathrm{q}\big(w_1 + w_2 - v\big) \leq \mathrm{q}\big(w_1 - v_1\big) + \mathrm{q}\big(w_2 - v_2\big) + \mathrm{q}\big(v_1 + v_2 - v\big) < \frac{r-\delta}{2} + \frac{r-\delta}{2} + \delta = r.$$

As $B \subseteq V \times V$ is open in the product topology as a product of open sets, this implies the continuity of the vector space addition. Similarly, for the multiplication by scalars, we construct open neighbourhoods of

$$(z_0, w_0) \in \cdot^{-1}\big(\mathrm{B}_{\mathrm{q},r}(v)\big) = \big\{(z, w) \in \mathbb{C} \times V \colon \mathrm{q}(zw - v) < r\big\}.$$

Indeed, if $(z, w) \in \mathrm{B}_{r_1}(z_0) \times \mathrm{B}_{\mathrm{q},r_2}(w_0)$ with $\delta := \mathrm{q}(zw - v)$, then

$$\begin{aligned}
\mathrm{q}(zw - v) &\leq \mathrm{q}(zw - zw_0) + \mathrm{q}(zw_0 - z_0 w_0) + \mathrm{q}(z_0 w_0 - v) \\
&= |z_0| \cdot \mathrm{q}(w - w_0) + |z - z_0| \cdot \mathrm{q}(w_0) + \mathrm{q}(z_0 w_0 - v) \\
&< |z_0| \cdot r_2 + r_1 \cdot \mathrm{q}(w_0) + \delta.
\end{aligned}$$

Hence, setting

$$r_1 \coloneqq r_2 \coloneqq \frac{r - \delta}{2(1 + \mathrm{q}(w_0) + |z_0|)}$$

implies

$$\mathrm{B}_{r_1}(z_0) \times \mathrm{B}_{\mathrm{q},r_2}(w_0) \subseteq \cdot^{-1}\big(\mathrm{B}_{\mathrm{q},r}(v)\big).$$

We have thus shown *i.)*. Now, *ii.)* is clear, as the inclusions $V \hookrightarrow V \times V$ are topological embeddings. For *iii.)*, let $\mathrm{q} \in \mathrm{cs}(V)$. Then, by definition of continuity, the preimage

$$\mathrm{q}^{-1}\big([0, 1)\big) \subseteq V$$

is open. Hence, it can be written as a union of open cylinders corresponding to elements of $\mathcal{P}$. In particular, we find $\mathrm{p} \in \mathcal{P}$ and $r > 0$ such that

$$\mathrm{B}_{\mathrm{p},r}(0) \subseteq \mathrm{q}^{-1}\big([0, 1)\big) = \mathrm{B}_{\mathrm{q},1}(0),$$

where we use $\mathrm{q}(0) = 0$.[1] Unwrapping the definition of the open cylinder, this means

$$\mathrm{p}(v) < r \quad \Longrightarrow \quad \mathrm{q}(v) < 1$$

for all $v \in V$. By homogeneity, we may rephrase this as the continuity estimate

$$\mathrm{q}(v) \leq \frac{1}{r} \cdot \mathrm{p}(v) \qquad \text{for all } v \in V.$$

This is (2.5), where we may thus even choose $n = 1$. Assume conversely (2.5) holds for some seminorm q and suitably chosen $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{P}$ and $c_1, \ldots, c_n > 0$. As $\mathcal{P}$ is filtrating, we may pass to a joint upper bound $\mathrm{p} \in \mathcal{P}$ of $\mathrm{p}_1, \ldots, \mathrm{p}_n$ to obtain

$$\mathrm{q} \leq c \cdot \mathrm{p}$$

with $c \coloneqq c_1 + \cdots + c_n$. Reversing the logic from before, this means

$$\mathrm{B}_{\mathrm{p},1}(0) \subseteq \mathrm{B}_{\mathrm{q},c}(0) = \mathrm{q}^{-1}\big([0, c)\big).$$

Hence, p is continuous at the origin. Let now $(v_\alpha)_{\alpha \in J}$ be a convergent net with limit $v \in V$. Then the net $(v - v_\alpha)_\alpha$ converges to zero by *ii.)*. By what we have already shown, given $\varepsilon > 0$ we thus find $\alpha_0 \in J$ such that

$$\mathrm{p}(v_\alpha) \leq \mathrm{p}(v) + \mathrm{p}(v - v_\alpha) \leq \varepsilon \qquad \text{for all } \alpha \succcurlyeq \alpha_0.$$

Bringing $\mathrm{p}(v)$ to the other side and varying ε implies $\mathrm{p}(v_\alpha) \to \mathrm{p}(v)$, establishing the continuity of p on all of V. This completes the proof of *iii.)*. We turn towards *iv.)*. By Definition 2.8, it is clear that the topology induced by $\mathrm{cs}(V)$ is finer than the one induced by $\mathcal{P}$. It thus suffices to prove that given $\mathrm{q} \in \mathrm{cs}(V)$, $r > 0$ and $v \in V$, the associated open cylinder $\mathrm{B}_{\mathrm{q},r}(v)$ is already a member of the topology generated by $\mathcal{P}$. For $v = 0$, this is just the geometric formulation of *iii.)*. Invoking *ii.)*, this implies the openness of

$$\tau_v\big(\mathrm{B}_{\mathrm{q},r}(0)\big) = \mathrm{B}_{\mathrm{q},r}(v)$$

[1] If you are not familiar with this type of reasoning within topological spaces, prove this!

as homeomorphisms are, in particular, open mappings. Part *v.)* is now immediate, as each $\mathrm{B}_{\mathrm{q},r}(v)$ for $\mathrm{q} \in \mathrm{cs}(V)$ contains an open cylinder associated to a seminorm from the defining system $\mathcal{P}$ by what we have just shown. Finally, assume the Hausdorff property. Then, we may separate $v \neq 0$ from the origin. In particular, there exist $\mathrm{q} \in \mathcal{P}$ and $r > 0$ with

$$v \notin \mathrm{B}_{\mathrm{q},r}(0),$$

which means $\mathrm{q}(v) > r > 0$. Conversely, assume the condition and let $v, w \in V$ with $v \neq w$. Then $v - w > 0$ and thus we may invoke our assumption to find $\mathrm{q} \in \mathrm{cs}(V)$ such that $r := \mathrm{q}(v - w) > 0$. Consequently,

$$\mathrm{B}_{\mathrm{q},r/2}(0) \cap \mathrm{B}_{\mathrm{q},r/2}(v - w) = \emptyset$$

by the triangle inequality. Translating by w via *ii.)* then yields

$$\mathrm{B}_{\mathrm{q},r/2}(w) \cap \mathrm{B}_{\mathrm{q},r/2}(v) = \emptyset.$$

Hence, V is Hausdorff in this case. □

In the proof of *iii.)*, we have seen that it suffices to consider the case $n = 1$. Our more general formulation is however often useful in practice. Not every topological vector space is locally convex.

Exercise 2.10 Let $p \in (0, 1)$ and $J \neq \emptyset$ an index set. Consider

$$\ell^p(J) := \Big\{ a \in \mathrm{Map}(J, \mathbb{C}) \colon \sum_{\alpha \in J} |a_\alpha|^p < \infty \Big\}.$$

Prove the following:

i.) The mapping $d \colon \ell^p \times \ell^p \longrightarrow [0, \infty)$,

$$d(a, b) := \sum_{\alpha \in J} \big| a_\alpha - b_\alpha \big|^p$$

defines a translation-invariant metric on ℓ^p.

ii.) The metric space (ℓ^p, d) is complete.

iii.) The vector space operations on (ℓ^p, d) are continuous, i.e. ℓ^p constitutes a topological vector space.

iv.) The mapping $a \mapsto d(a, 0)$ is not a norm.

Finally, let $J := \{0, 1\}$. Sketch the metric balls around the origin. Conclude $\ell^p(J)$ is typically not locally convex.

The following exercise ties a loose end from earlier.

Exercise 2.11 Let V be locally convex, $\mathrm{q} \in \mathrm{cs}(V)$ and $r > 0$. Prove that the open cylinder $\mathrm{B}_{\mathrm{q},r}(v)$ is sequentially dense in the closed cylinder

$$\mathrm{B}_{\mathrm{q},r}(v)^{\mathrm{cl}} := \big\{ w \in V \colon \mathrm{q}(v - w) \leq r \big\}.$$

This, a posteriori, justifies our notation.

Next, we take a brief look at how convergence within a locally convex space may be described. Ultimately, the generalization is both straightforward and convenient.

Lemma 2.12 *Let V be a locally convex space with defining system of seminorms $\mathcal{P}$. Then a net $(v_\alpha)_{\alpha \in J}$ converges to a vector $v \in V$ iff for every $\mathrm{q} \in \mathcal{P}$ and every $\varepsilon > 0$, there exists an index $\alpha_0 \in J$ such that*

$$\mathrm{q}\big(v_\alpha - v\big) \le \varepsilon \qquad \textit{for all } \alpha \succcurlyeq \alpha_0. \tag{2.6}$$

In this case, the condition holds for all $\mathrm{q} \in \mathrm{cs}(V)$.

PROOF: Assume first that the net $(v_\alpha)_{\alpha \in J}$ is convergent with limit $v \in V$ and let $\mathrm{q} \in \mathrm{cs}(V)$. Then the open cylinder $\mathrm{B}_{\mathrm{q},\varepsilon}(v)$ constitutes an open neighbourhood of v by Proposition 2.9,*iv.)* and *v.)*. By convergence of $(v_\alpha)_{\alpha \in J}$ there thus is some $\alpha_0 \in J$ such that $v_\alpha \in \mathrm{B}_{\mathrm{q},\varepsilon}(v)$ for all $\alpha \succcurlyeq \alpha_0$. This is precisely (2.6). Conversely, let again $\varepsilon > 0$ as well as $\mathrm{q} \in \mathcal{P}$ and assume the validity of (2.6). Then $v_\alpha \in \mathrm{B}_{\mathrm{q},\varepsilon}(v)$ for all $\alpha \succcurlyeq \alpha_0$. As the open cylinders corresponding to seminorms from the defining system form a basis of the topology, this already proves the convergence of (v_α) to v. □

Motivated by the Lemma, we call a net $(v_\alpha)_{\alpha \in J} \subseteq V$ within a locally convex space V Cauchy if for every $\mathrm{q} \in \mathcal{P}$ and every $\varepsilon > 0$ there exists an index $\alpha_0 \in J$ such that

$$\mathrm{q}\big(v_\alpha - v_\beta\big) \le \varepsilon \qquad \text{for all } \alpha, \beta \succcurlyeq \alpha_0.$$

In this case, the condition holds for all $\mathrm{q} \in \mathrm{cs}(V)$. As usual, we then call V complete if all Cauchy nets converge. If the topology of V actually arises from a single norm, then this indeed recovers the usual notions of Cauchy nets and, by first countability, of completeness.

Exercise 2.13 Let V be a vector space and $\|\cdot\|$ a norm on V. Prove that the locally convex topology induced by $\mathcal{P} := \{\|\cdot\|\}$ is the norm-topology. Can you describe the set of all continuous seminorms $\mathrm{cs}(V)$ on V?

Each of the systems of seminorms from Example 2.2 now induces the corresponding vector spaces with locally convex topologies.

Exercise 2.14 Let J be a set and consider the space of complex sequences

$$\mathrm{Map}(J, \mathbb{C}) := \big\{a \colon J \longrightarrow \mathbb{C}\big\}.$$

i.) Construct a defining system of seminorms such that the resulting locally convex topology is the topology of pointwise convergence.

Hint: Remember that defining systems are filtrating.

ii.) Establish the completeness of the locally convex space $\mathrm{Map}(J, \mathbb{C})$.

iii.) Let $\|\cdot\|$ be a norm on $\mathrm{Map}(J, \mathbb{C})$ and assume that J is infinite. Prove that $\|\cdot\|$ is discontinuous.

iv.) Show that the space

$$c_{00}(J) := \big\{a \in \mathrm{Map}(J, \mathbb{C}) \colon \exists_{F \subseteq \mathcal{F}} \forall_{\alpha \notin F}\, a_\alpha = 0\big\}. \tag{2.7}$$

of compactly supported sequences is dense within $\mathrm{Map}(J, \mathbb{C})$. Here $\mathcal{F}$ denotes the collection of finite subsets of J.

Hint: It is *not* sequentially dense for uncountable J. Use nets.

Remark 2.15 Let V be a locally convex space. Replacing finite sets with finite dimensional subspaces, one may prove that $\widehat{V'_\sigma} = V^*$, i.e. every linear functional on V is the pointwise net limit of continuous linear functionals. In particular, V'_σ is typically incomplete. It does, however, fulfil the weaker property of being quasi-complete. That is to say, bounded Cauchy nets converge. This is sufficient for many applications, as this guarantees the existence of e.g. Gelfand-Pettis integrals.

Proposition 2.16 *Let $L\colon V \longrightarrow W$ be a linear mapping between locally convex spaces V and W with corresponding defining systems of seminorms $\mathcal{P}_V$ and $\mathcal{P}_W$. Then the following are equivalent:*

i.) The map L is uniformly continuous.

ii.) The map L is continuous.

iii.) The map L is continuous at some $v \in V$.

iv.) The map L is continuous at the origin.

v.) For every $\mathrm{q} \in \mathrm{cs}(V)$ there exist $\mathrm{p} \in \mathcal{P}_V$ and $c > 0$ such that

$$\mathrm{q}\big(Lv\big) \leq c \cdot \mathrm{p}(v) \qquad \textit{for all } v \in V. \tag{2.8}$$

vi.) For every $\mathrm{q} \in \mathcal{P}_W$ there exists $\mathrm{p} \in \mathrm{cs}(V)$ such that

$$\mathrm{q}\big(Lv\big) \leq \mathrm{p}(v) \qquad \textit{for all } v \in V.$$

PROOF: Clearly, *i.)* implies *ii.)*, which in turn implies *iii.)*. Assuming L is continuous at a point $v \in V$ implies the continuity of the composition

$$w \quad \mapsto \quad \big(\tau_{-Lv} \circ L \circ \tau_v\big)(w) = L(v) - L(v+w) = Lw$$

at $w = 0$, as $\tau_v(0) = v$ and by Proposition 2.9, *ii.)*. This is *iv.)*. Alternatively, this follows readily by checking the continuity at zero by means of nets and using the linearity. Assume now *iv.)* and let $\mathrm{q} \in \mathrm{cs}(V)$. As $\mathrm{B}_{\mathrm{q},1}(0)$ constitutes an open neighbourhood of zero, the continuity yields the openness of its preimage under L. As the open cylinders centered at zero corresponding to $\mathcal{P}_V$ form a neighbourhood basis, we thus find $\mathrm{p} \in \mathcal{P}_V$ and $c > 0$ such that

$$\mathrm{B}_{\mathrm{p},1/c}(0) \subseteq L^{-1}\big(\mathrm{B}_{\mathrm{q},1}(0)\big).$$

Unwrapping this inclusion yields precisely (2.8). As multiples of continuous seminorms are continuous, *v.)* implies *vi.)*. Finally, assume *vi.)* and let $\mathrm{q} \in \mathcal{P}_W$ as well as $r > 0$. By rescaling our assumption, we find $\mathrm{p} \in \mathrm{cs}(V)$ such that

$$\mathrm{q}\big(Lv\big) \leq \mathrm{p}(v) \qquad \text{for all } v \in V.$$

That is to say,

$$Lv - Lw = L(v-w) \in \mathrm{B}_{\mathrm{q},r}(0) \qquad \text{for all } v, w \in V \text{ such that } v - w \in \mathrm{B}_{\mathrm{p},r}(0).$$

This is precisely the uniform continuity of L. □

Both *v.)* and *vi.)* are useful in practice: The former for using, the latter for checking continuity. It is instructive to apply Proposition 2.16 to some simple examples. They also illustrate the usefulness of working with small defining systems, while also being aware of many continuous seminorms.

Exercise 2.17 Let V be a finite dimensional locally convex Hausdorff space. Prove that, choosing a basis $(e_1, \ldots, e_d)$ of V, the coordinate mapping

$$\Phi\colon \mathbb{C}^d \longrightarrow V, \quad \Phi(z^1, \ldots, z^d) := \sum_{n=1}^{d} z^n e_n$$

constitutes a linear homeomorphism.

Exercise 2.18 Prove the continuity of the differentiation operator

$$\mathrm{D}\colon \mathscr{H}(\mathbb{C}) \longrightarrow \mathscr{H}(\mathbb{C}), \quad \mathrm{D} f := f'$$

and the multiplication operator

$$\mathrm{M}\colon \mathscr{H}(\mathbb{C}) \longrightarrow \mathscr{H}(\mathbb{C}), \quad \big(\mathrm{M} f\big)(z) := z f(z).$$

Compute also the commutator

$$\big[\mathrm{D}, \mathrm{M}\big] := \mathrm{D} \circ \mathrm{M} - \mathrm{M} \circ \mathrm{D}\,.$$

Conclude again that the topology of $\mathscr{H}(\mathbb{C})$ can not be induced by a norm.

Hint: It might be convenient to take another look at Exercise 2.3 in light of the new technology we have established in the meantime.

The situation, in which we find a countable defining system of seminorms deserves particular attention. In this case, the locally convex topology is first countable, i.e. every point possesses a countable basis of neighbourhoods.

Proposition 2.19 (Metrizability) *Let V be a locally convex space. Then the following are equivalent:*

i.) The space V is first countable.

ii.) There exists a countable defining system of seminorms for V.

iii.) There exists an ascending sequence (q_n) of seminorms defining the topology of V.

iv.) The topology of V is metrizable by means of a translation invariant metric d, i.e.

$$d(v + x, w + x) = d(v, w) \qquad \textit{for all } v, w, x \in V.$$

v.) The topology of V is metrizable.

Proof: Assuming *ii.)*, there exists a countable neighbourhood basis $(U_n)_n$ of the origin. As the open cylinders centered at 0 corresponding to continuous seminorms on V constitute a neighbourhood basis themselves, we find corresponding seminorms $\mathrm{p}_n \in \mathrm{cs}(V)$ such that

$$\mathrm{B}_{\mathrm{p}_n,1}(0) \subseteq U_n \qquad \text{for all } n \in \mathbb{N}_0.$$

Translating these balls by means of Proposition 2.9, *ii.)* we get countable neighbourhood bases consisting of open cylinders corresponding to the p_n. Thus, $\mathcal{P} := \{p_n : n \in \mathbb{N}_0\}$ constitutes a countable defining system of seminorms for the locally convex space V.

As $\mathrm{cs}(V)$ is closed under taking pointwise maxima of finitely many elements, we may define

$$\mathrm{q}_n := \max\{\mathrm{p}_1, \ldots, \mathrm{p}_n\} \qquad \text{for all } n \in \mathbb{N}_0$$

to obtain an ascending countable defining system of seminorms for V. Mimicking the construction from Example 1.1, this in turn yields a translation invariant metric d on V defined by

$$d(v, w) := \sum_{n=0}^{\infty} 2^{-n} \cdot \frac{\mathrm{q}_n(v - w)}{1 + \mathrm{q}_n(v - w)}.$$

Repeating our considerations regarding the metric balls establishes that d generates the topology of V. Finally, metrizable spaces are certainly first countable. □

If one (and thus all) of the conditions within Proposition 2.19 are fulfilled, and V is moreover complete Hausdorff, then we call V a Fréchet space. Banach spaces are particular examples of Fréchet spaces by Exercise 2.13. Roughly speaking, first countability ensures that the topology may be described by means of sequential concepts.

Exercise 2.20 Let V be a first countable topological space and $(v_\alpha)_{\alpha \in J}$ a convergent net. Prove that $(v_\alpha)_\alpha$ possesses a convergent subnet that is also a subsequence.

Hint: Subnets indexed by another index set I are induced by mappings $\phi\colon I \longrightarrow J$ such that for every index $\alpha \in J$ there exists another index $\beta \in I$ such that $\beta \preccurlyeq \beta'$ implies $\alpha \preccurlyeq \phi(\beta')$. A subnet is a subsequence if $I = \mathbb{N}$ and ϕ is strictly monotonic. Use a countable ordered basis of neighbourhoods of the limit as the index set.

This topological observation has many pleasant consequences for first countable locally convex spaces.

Exercise 2.21 Let V be a first countable locally convex space. Show the following:

i.) A subset $A \subseteq V$ is closed iff it is sequentially closed.

ii.) A linear mapping $L\colon V \longrightarrow W$ with values in another locally convex space is continuous iff it is sequentially continuous.

iii.) The space V is complete iff it is sequentially complete.

As the First Baire Category Theorem applies to complete metric spaces, it should be of no surprise that there are Fréchet space generalizations of the Banach-Steinhaus Theorem, the Open Mapping Theorem and the Closed Graph Theorem. In fact, appropriately phrased, they remain valid beyond the setting of Fréchet spaces. The exploration of these topics however goes way beyond the scope of these lecture notes. Discussions can be found in any of the excellent textbooks mentioned at the beginning of the text, where we once again highlight [11, Ch. 4 & 5] for a particularly pleasant experience.

Instead, we proceed with a locally convex incarnation of the Hahn-Banach Extension Theorem, which we shall need for our exploration of tensor products.

Theorem 2.22 (Hahn-Banach) *Let V be a vector space and* $\mathrm{q}\colon V \longrightarrow \mathbb{R}$ *be sublinear. Furthermore, let $U \subseteq V$ be a subspace and*

$$u'\colon U \longrightarrow \mathbb{C}$$

be a linear functional on U such that $|u'| \leq \mathrm{q}$. Then there exists a linear functional $v'\colon V \longrightarrow \mathbb{C}$ such that

$$v'\Big|_U = u' \qquad \text{and} \qquad |v'| \leq \mathrm{q}.$$

In view of Proposition 2.16, this means that we may extend continuous linear functional from subspaces in manner that preserves continuity estimates. This statement is ultimately a purely algebraic application of Zorn's Lemma. It is quite likely that you have already seen a sufficiently general incarnation before. Thus, instead of rehearsing the details, we conclude our abstract considerations with a quite useful consequence for later.

Exercise 2.23 Let V be locally convex and $\mathrm{q} \in \mathrm{cs}(V)$. Then

$$\mathrm{q}(v) = \sup_{|v'| \leq \mathrm{q}} \big|v'(v)\big| \qquad \text{for all } v \in V. \tag{2.9}$$

Another amusing application is the existence of so-called Banach limits.

Exercise 2.24 (Banach Limits) Let $L\colon \ell^\infty(\mathbb{R}) \longrightarrow \ell^\infty(\mathbb{R})$ be the left-shift on the space of real bounded sequences, i.e. $(La)_n \coloneqq a_{n+1}$ for all $a \in \ell^\infty(\mathbb{R})$. Prove that there exists a continuous linear functional Λ on ℓ^∞ such that

$$\Lambda(La) = \Lambda(a) \qquad \text{and} \qquad \liminf_{n\to\infty} a_n \leq \Lambda(a) \leq \limsup_{n\to\infty} a_n$$

for all $a \in \ell^\infty(\mathbb{R})$.

Hint: Consider the functionals

$$\Lambda_n(a) \coloneqq \frac{1}{n} \sum_{k=0}^{n-1} a_k$$

defined on the subspace

$$Y \coloneqq \big\{a \in \ell^\infty(\mathbb{R})\colon \lim_{n\to\infty} \Lambda_n(a) \text{ exists}\big\}.$$

In Exercise 2.10, we have seen that the topological vector spaces ℓ^p with $p \in (0,1)$ are not locally convex. Nevertheless, it makes sense to think about their continuous dual.

Exercise 2.25 Let $p, q \in (0,\infty) \cup \{\infty\}$. Prove the following statements:

i.) If $p < q$, then $\ell^p \subsetneq \ell^q$.

ii.) If $p \in (0,1]$, then $(\ell^p)' \cong \ell^\infty$ via the bijective linear isometry

$$\Phi\colon \ell^\infty \longrightarrow (\ell^p)', \quad \Phi(a)\gamma \coloneqq \sum_{n=0}^\infty a_n \cdot \gamma_n.$$

While this is already rather jarring, Lebesgue spaces behave much worse.

Exercise 2.26 Let $p \in (0,1)$ and consider the Lebesgue space $L^p([0,1])$ of absolutely p-integrable equivalence classes of measurable functions on $[0,1]$. Show the following:

i.) If $p < q < 1$, then $L^q([0,1]) \subsetneq L^p([0,1]) \subsetneq L^1([0,1])$.

ii.) The translation-invariant metric

$$d(f, g) := \int_0^1 \big|f(x) - g(x)\big|^p \, \mathrm{d}x$$

endows $L^p([0,1])$ with a topological vector space structure.

iii.) The convex hull of any ball centered at the origin is all of $L^p([0,1])$.

Hint: Fix $f \in L^p([0,1])$, partition the unit interval into $n \in \mathbb{N}$ subintervals in a suitable manner.

iv.) The convex hull of any non-empty open subset of $L^p([0,1])$ is all of $L^p([0,1])$.

v.) Conclude that $L^p([0,1])$ is bounded. Conclude further that the continuous dual of $L^p([0,1])$ is trivial.

Hint: Establish first that continuous linear mappings preserve bounded subsets.

Combining both spaces results in the following.

Exercise 2.27 Let $p \in (0,1)$. Prove the following:

i.) The metric space $L^p([0,1])$ is separable, i.e. possesses a dense countable subset.

Hint: It suffices to exhibit a dense vector subspace of countable dimension. Why?

ii.) If $(f_n)_n \subseteq L^p([0,1])$ is dense within the metric unit ball, then linearly extending

$$Le_n := f_n$$

results in a continuous operator $L\colon c_{00}(\mathbb{N}_0) \longrightarrow L^p([0,1])$, where $c_{00}(\mathbb{N}_0)$ is defined as in (2.7) and endowed with the ℓ^p-topology.

iii.) Conclude that L extends to a surjective continuous linear mapping

$$L\colon \ell^p(\mathbb{N}_0) \longrightarrow L^p([0,1]).$$

iv.) If $x' \in (\ell^p)'$ is a continuous linear functional with

$$x'\Big|_{\ker L} \equiv 0,$$

then $x' = 0$. What does this mean for the continuous dual of the (indeed well-defined Hausdorff topological vector space) quotient $\ell^p/\ker T$?

Having convinced ourselves that general topological vector spaces house eldritch horrors beyond our comprehension, we return into the soothing comfort of the locally convex world and construct the Hausdorffization functor. The principal idea is that by Proposition 2.9, *vi.)*, the failure of the Hausdorff property of a locally convex space V with defining system $\mathcal{P}$ of seminorms lies precisely in the intersection of kernels

$$V_0 := \bigcap_{\mathrm{q} \in \mathrm{cs}(V)} \ker \mathrm{q} = \bigcap_{\mathrm{q} \in \mathcal{P}} \ker \mathrm{q}, \tag{2.10}$$

which constitutes a closed subspace of V. We are thus lead to consider the quotient V/V_0, which we endow with the final topology corresponding to the quotient projection, as usual. Unlike general final topologies, quotients always remain within the category.

Lemma 2.28 *Let V be a locally convex space and $X \subseteq V$ a subspace.*

i.) The quotient V/X is locally convex. More precisely, if $\mathcal{P}$ is a defining system of seminorms for V, then

$$[\mathcal{P}] := \{[\mathrm{q}] : \mathrm{q} \in \mathcal{P}\}$$

with

$$[\mathrm{q}]\big([v]\big) := \inf_{w \in [v]} \mathrm{q}(w) \quad \textit{for all } v \in V \tag{2.11}$$

constitutes a defining system of seminorms for V/X.

ii.) If V is Hausdorff and X is closed, then V/X is Hausdorff.

Proof: Perhaps the simplest way of arguing is by means of Proposition 2.5 combined with general topological considerations. Indeed, as we are dividing by the action of the abelian group X on V, the quotient projection $\pi\colon V \longrightarrow V/X$ is open: If $U \subseteq V$ is open, then so is

$$\pi^{-1}\big(\pi(U)\big) = \bigcup_{x \in X} \{u + x : u \in U\} = \bigcup_{x \in X} \tau_x(U)$$

as the union of open sets, where we use that the translations by $x \in X$ constitute homeomorphisms. Now, by definition of the final topology, a subset of the quotient X/V is open iff its preimage under π is open. Thus $\pi(U)$ is open, establishing the openness of π. Thus, by linearity,

$$U_r := \pi\big(\mathrm{B}_{\mathrm{q},r}(0)\big) \subseteq V/X$$

is an absolutely convex as well as absorbing open neighbourhood of zero for all $\mathrm{q} \in \mathcal{P}$ and $r > 0$. Invoking Proposition 2.5 to pass back to the corresponding Minkowski functionals and varying the radius $r > 0$, we obtain the local convexity of X/V. Moreover, plugging in the definition of the Minkowski functional from (2.2) combined with (2.3) yields

$$\begin{aligned}
\mathrm{p}_{U_r}([v]) &= \inf\big\{s > 0 \;\big|\; [v] \in s \cdot \pi\big(\mathrm{B}_{\mathrm{q},r}(0)\big)\big\} \\
&= \inf\big\{s > 0 \;\big|\; [v] \in \pi\big(\mathrm{B}_{\mathrm{q},rs}(0)\big)\big\} \\
&= \inf\big\{s > 0 \;\big|\; \exists_{w \in [v]} : w \in \mathrm{B}_{\mathrm{q},rs}(0)\big\} \\
&= \inf_{w \in [v]} \mathrm{p}_{\mathrm{B}_{\mathrm{q},r}(0)}(w) \\
&= \big[\mathrm{p}_{\mathrm{B}_{\mathrm{q},r}(0)}\big]\big([v]\big)
\end{aligned}$$

for all $\mathrm{q} \in \mathcal{P}$, $r > 0$ and $v \in V$. This completes the proof of the first part. For the second, we once again view V/X as a quotient by a group action. Note first that the equivalence relation

$$\sim\, = \big\{(v, v + x) : v \in V, x \in X\big\} \subseteq V \times V$$

is closed. Indeed, if $(v_\alpha, v_\alpha + x_\alpha) \subseteq \sim$ is a convergent net with limit $(v, w) \in V \times V$, then

$$x_\alpha = (v_\alpha + x_\alpha) - v_\alpha \longrightarrow w - v \in X$$

by continuity of the vector space operations and closedness of X. Hence,

$$w = v + \lim x_\alpha \in [v]$$

as desired. Let now $v \in V$ with $[v] \neq [0]$, i.e. $(v, 0) \notin \sim$. By openness of the complement of $\sim$, we thus find open sets $U_1, U_2 \subseteq V$ with $v \in U_1$ and $0 \in U_2$ such that

$$U_1 \times U_2 \not\subseteq \sim .$$

We check that $\pi(U_1) \cap \pi(U_2) = \emptyset$. Indeed, if $[w] \in \pi(U_1) \cap \pi(U_2)$, then

$$w \in \pi^{-1}\big(\pi(U_1) \cap \pi(U_2)\big) = \pi^{-1}\big(\pi(U_1)\big) \cap \pi^{-1}\big(\pi(U_2)\big) = \bigcup_{x \in X} \tau_x(U_1) \cup \bigcup_{x \in X} \tau_x(U_2)$$

That is to say, w is in the orbit of both v and 0, which is a contradiction to $(v, 0) \notin \sim$ by virtue of the transitivity of the equivalence relation. $\square$

This lets us define the Hausdorffization functor on objects V as V/V_0 with V_0 from (2.10). Remarkably, continuity estimates of the form (2.8) force compatibility with the quotient by V_0.

Lemma 2.29 *Let $L\colon V \longrightarrow W$ be a continuous linear mapping between locally convex spaces with closed subspaces. Then there exists a unique continuous linear mapping*

$$L_0\colon V/V_0 \longrightarrow W/W_0$$

such that $L_0 \circ \pi_V = \pi_W \circ L$, where $\pi_V\colon V \longrightarrow V/V_0$ and $\pi_W\colon W \longrightarrow W/W_0$ denote the quotient projections.

PROOF: We apply the characteristic property of the quotient V/V_0 to L. To this end, let $v \in V_0$ and $\mathrm{q} \in \mathrm{cs}(W)$. Invoking the characterization of continuity from Proposition 2.16, *v.)*, we find a corresponding $\mathrm{p} \in \mathrm{cs}(V)$ such that

$$0 \leq \mathrm{q}\big(Lv\big) \leq \mathrm{p}(v) = 0$$

by virtue of $v \in V_0$. Hence, $v \in \ker(\pi_W \circ L)$, which implies $V_0 \subseteq \ker(\pi_W \circ L)$. Consequently, the characteristic property of V/V_0, viewed as a vector space quotient, yields a unique linear mapping

$$L_0\colon V/V_0 \longrightarrow W/W_0 \qquad \text{such that} \qquad L_0 \circ \pi_V = \pi_W \circ L.$$

As we have endowed V/V_0 with the final topology with respect to π_V, the continuity of L_0 follows from the continuity of $L_0 \circ \pi_V = \pi_W \circ L$. $\square$

Putting both results together now yields the promised functor. We denote the category of locally convex spaces with continuous linear mappings as morphisms by lcs. Adding the Hausdorff property to the objects yields a full subcategory Lcs.

Corollary 2.30 *The assignment*

$$\mathfrak{H}\colon \mathsf{lcs} \longrightarrow \mathsf{Lcs}, \qquad \begin{cases} \mathfrak{H}(V) \coloneqq V_0, \\ \mathfrak{H}\big(V \xrightarrow{L} W\big) \coloneqq \big(V/V_0 \xrightarrow{L_0} W/W_0\big) \end{cases}$$

constitutes a covariant functor.

Proof: Let V be a locally convex space with identity mapping $\mathrm{id}_V \colon V \longrightarrow V$. Then

$$\mathrm{id}_{V/V_0} \circ \pi_V = \pi_V = \pi_V \circ \mathrm{id}_V$$

and thus $\mathfrak{H}(\mathrm{id}_V) = \mathrm{id}_{V/V_0}$ by virtue of the uniqueness within Lemma 2.29. Let now

$$V \xrightarrow{L} W \xrightarrow{T} X$$

be continuous linear mappings between locally convex spaces with induced mappings

$$V/V_0 \xrightarrow{L_0} W/W_0 \xrightarrow{T_0} X/X_0$$

from Lemma 2.29. Then

$$\big(T_0 \circ L_0\big) \circ \pi_V = T_0 \circ \pi_W \circ L = \pi_X \circ \big(T \circ L\big)$$

and thus $(T \circ L)_0 = T_0 \circ L_0$ by uniqueness. This completes the proof. □

We call $\mathfrak{H}$ the Hausdorffization functor. While this construction may seem rather abstract, we are actually familiarized ourselves with its very essence in the setting of measure theory.

Example 2.31 Let (Ω, μ) be a measure space. Consider

$$\mathscr{M}(\Omega) := \big\{ f \in \mathrm{Map}(\Omega, \mathbb{C}) \colon f \text{ is measurable} \big\}$$

as well as

$$\mathcal{L}^1(\Omega) := \Big\{ f \in \mathscr{M}(\Omega) \colon \|f\|_1 := \int_\Omega \big|f\big| \,\mathrm{d}\mu < \infty \Big\}.$$

Then the Hausdorffization

$$L^1(\Omega) := \mathcal{L}^1(\Omega) / \ker \|\cdot\|_1$$

is the usual Lebesgue space consisting of almost-everywhere equivalence classes.

After our abstract considerations on locally convex spaces, our goal is now to endow the space of complex-valued smooth functions $\mathscr{C}^\infty(M)$ defined on a manifold M with a locally convex topology. In Example 2.2, *iii.)*, we have already met seminorms for smooth functions defined on open subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$. As usual in differential geometry, the idea is then that this reflects the situation within a chart. While this approach is perfectly serviceable, there is a more conceptual (and ultimately equivalent) approach based on differential operators and having already established the $\mathscr{C}$-topology, see again Example 1.1.

Definition 2.32 ($\mathscr{C}$-topology) *Let M be a topological space. Then the $\mathscr{C}$-topology on the space of complex-valued continuous functions $\mathscr{C}(M)$ is the locally convex topology generated by the seminorms*

$$\mathrm{q}_K(f) := \max_{p \in K} \big| f(p) \big|,$$

where we vary $K \subseteq M$ through the compact subsets of M.

As Recall Grothendieck's [4] recursive definition of differential operators of an associative algebra $\mathscr{A}$ as it is e.g. discussed in [9, Ch. 15]. That is, $\operatorname{DiffOp}(\mathscr{A})$ is the filtered algebra

$$\begin{aligned} &\operatorname{DiffOp}^0(\mathscr{A}) := \big\{M_a \colon a \in \mathscr{A}\big\}, \\ &\operatorname{DiffOp}^k(\mathscr{A}) := \big\{D \in \mathrm{L}(\mathscr{A}) \bigm| \forall_{a\in\mathscr{A}} \colon [D, M_a] \in \operatorname{DiffOp}^{k-1}(\mathscr{A})\big\} \qquad \text{for } k > 0, \end{aligned}$$

where $\mathrm{L}(\mathscr{A})$ is the space of $\mathbb{k}$-linear maps from $\mathscr{A}$ to $\mathscr{A}$, the M_a are multiplication operators with $a \in \mathscr{A}$ and

$$\big[\,\cdot\,,\,\cdot\,\big] \colon\ \mathrm{L}(\mathscr{A}) \times \mathrm{L}(\mathscr{A}) \longrightarrow \mathrm{L}(\mathscr{A}), \quad [D, D'] := D \circ D' - D' \circ D$$

is the commutator induced from the associative algebra structure of $\mathrm{L}(\mathscr{A})$. If $\mathscr{A}$ is unital, then

$$\operatorname{DiffOp}^0(\mathscr{A}) \cong \operatorname{End}_{\mathscr{A}}(\mathscr{A}) \cong \mathscr{A},$$

where $\operatorname{End}_{\mathscr{A}}(\mathscr{A})$ denotes the set of $\mathscr{A}$-linear endomorphisms of $\mathscr{A}$. Moreover,

$$\operatorname{DiffOp}^1(\mathscr{A}) \cong \operatorname{End}_{\mathscr{A}}(\mathscr{A}) \oplus \operatorname{Der}_{\mathscr{A}},$$

where $\operatorname{Der}(\mathscr{A})$ denotes the space of all linear derivations on $\mathscr{A}$. The algebra we are interested in is of course $\mathscr{A} = \mathscr{C}^\infty(M)$, where we get

$$\operatorname{DiffOp}^1(M) := \operatorname{DiffOp}^1\big(\mathscr{C}^\infty(M)\big) \cong \operatorname{Der}\big(\mathscr{C}^\infty(M)\big) \cong \Gamma^\infty(TM) \tag{2.12}$$

are the vector fields. More generally, one may consider $\mathbb{C}$-linear mappings

$$D \colon \mathscr{E} \longrightarrow \mathscr{F}$$

between unital $\mathscr{A}$-modules $\mathscr{E}$ and $\mathscr{F}$. Accordingly, the multiplication operators M_a with $a \in \mathscr{A}$ now act on $\mathscr{E}$ and $\mathscr{F}$ by module multiplication. Hence, we may also speak of differential operators

$$D \colon \Gamma^\infty(E) \longrightarrow \Gamma^\infty(F)$$

between spaces of sections of vector bundles $\mathrm{pr}_E \colon E \longrightarrow M$ and $\mathrm{pr}_F \colon F \longrightarrow M$ over the same base manifold M. With these preliminaries, we may define the $\mathscr{C}^k$-topologies for finite k. The idea is that we may view a differential operator of order k as a linear mapping

$$D \colon \mathscr{C}^k(M) \longrightarrow \mathscr{C}(M). \tag{2.13}$$

To see this, one may combine the recursive definition with (2.12). Alternatively, one has to fall back on the usual localized description of differential operators within a chart. That being said, whatever the $\mathscr{C}^k$-topology may entail, we certainly want all mappings (2.13) to be continuous. As we are already in possession of a reasonable topology on $\mathscr{C}(M)$, the natural choice thus becomes the initial topology, i.e. the coarsest topology such that all mappings (2.13) become continuous.

Definition 2.33 ($\mathscr{C}^k$-topology) *Let M be a manifold and $k \in \mathbb{N}$. The $\mathscr{C}^k$-topology on the space of k-times differentiable functions $\mathscr{C}^k(M)$ is the initial topology with respect to the mappings*

$$D \colon \mathscr{C}^k(M) \longrightarrow \mathscr{C}(M),$$

where we vary $D \in \operatorname{DiffOp}^k(M)$.

The definition for the Γ^k-topology of k-times differentiable sections is mutatis mutandis identical. Remarkably, initial topologies induced by linear mappings into locally convex spaces are automatically locally convex.

Proposition 2.34 *Let*

$$L_\alpha \colon V \longrightarrow V_\alpha$$

be linear mappings from a vector space V with values in locally convex spaces V_α. Then the associated initial topology on V is locally convex. More precisely, if $\mathcal{P}_\alpha$ are defining systems of seminorms for V_α for all $\alpha \in J$, then

$$\mathcal{P} := \left\{ \sum_{\alpha \in F} L_\alpha^* \mathrm{p}_\alpha := \mathrm{p}_\alpha \circ L_\alpha \colon F \subseteq J \text{ finite}, \mathrm{p}_\alpha \in \mathcal{P}_\alpha \right\} \tag{2.14}$$

constitutes a defining system of seminorms for V.

PROOF: By linearity of the ϕ_α, the pullbacks $\phi_\alpha^* \mathrm{p}_\alpha$ are indeed seminorms on V. As locally convex topologies are defined by means of the bases (2.4), we get a basis for the topology of V, which is given by

$$\mathcal{B} := \left\{ L_\alpha^{-1}\big(\mathrm{B}_{\mathrm{p}_\alpha, r}(v_\alpha)\big) \colon \alpha \in J, \mathrm{p}_\alpha \in \mathcal{P}_\alpha, r > 0, v_\alpha \in V_\alpha \right\}.$$

Notice now

$$\phi_\alpha^{-1}\big(\mathrm{B}_{\mathrm{p}_\alpha, r}(L_\alpha v)\big) = \left\{ w \in V \colon \mathrm{p}_\alpha\big(L_\alpha v - L_\alpha w\big) < r \right\} = \mathrm{B}_{\phi_\alpha^* \mathrm{p}_\alpha, r}(v)$$

for all $w \in V$, $\alpha \in J$ and $r > 0$. Now, if $v_\alpha \notin L_\alpha V$, then

$$\phi_\alpha^{-1}\big(\mathrm{B}_{\mathrm{p}_\alpha, r}(L_\alpha v)\big) = \emptyset$$

for all sufficiently small $r > 0$. Thus, as continuity is a local property, we may simplify $\mathcal{B}$ to

$$\mathcal{B}' := \left\{ \mathrm{B}_{\phi_\alpha^* \mathrm{p}_\alpha, r}(v) \colon \alpha \in J, \mathrm{p}_\alpha \in \mathcal{P}_\alpha, r > 0, v \in V \right\},$$

which establishes the local convexity of V with defining system of seminorms as described within (2.14). Note that we have taken finite sums to make $\mathcal{P}$ filtrating. $\square$

Returning to the concrete situation, we thus get the following.

Corollary 2.35 *Let M be a manifold and $k \in \mathbb{N}$. Then the $\mathscr{C}^k$-topology on $\mathscr{C}^k(M)$ is locally convex with defining system of seminorms given by*

$$\mathrm{p}_{K,D}(f) = \max_{p \in K} \big| Df \big|, \tag{2.15}$$

where we vary $K \subseteq M$ through the compact subsets of M and $D \in \mathrm{DiffOp}^k(M)$.

Plugging in the local form of differential operators now in principle allows us to return to the chart-based approach we have indicated earlier. Indeed, if (U, x) is a chart of M and $K \subseteq M$ is compact, then localizing $D_U = D^\alpha \partial_\alpha$ yields the continuous seminorm

$$\mathrm{r}_{U,K,D}(f) := \max_{p \in K} \left| D^\alpha \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right|$$

on $\mathscr{C}^k(M)$. Here, we employ the usual multi-index notation and also Einstein's summation convention. By our considerations, the locally convex topology arising from such seminorms is independent of the chosen atlas and instead intrinsic to the smooth structure.

Finally, the passage to $\mathscr{C}^\infty(M)$ is based on the simple observation that

$$\mathscr{C}^\infty(M) \hookrightarrow \cdots \hookrightarrow \mathscr{C}^{k+1}(M) \hookrightarrow \mathscr{C}^k(M) \hookrightarrow \mathscr{C}^{k-1}(M) \hookrightarrow \cdots,$$

i.e. we have canonical linear injections $\mathscr{C}^\infty(M) \hookrightarrow \mathscr{C}^k(M)$ for all $k \in \mathbb{N}_0$. Hence, we may once again use the initial topology to endow $\mathscr{C}^\infty(M)$ with its own locally convex topology. Taking another look at (2.14), the continuous seminorms are simply given by (2.15), where $D \in \operatorname{DiffOp}(M)$ is now any differential operator of M. Note that there are still only *finitely* many derivatives involved in each seminorm. Having established this point of view, the following structural observations are easy but nevertheless instructive.

Proposition 2.36 *Let M be a manifold and $k \in \mathbb{N} \cup \{\infty\}$.*

i.) The $\mathscr{C}^k$-topology on $\mathscr{C}^k(M)$ is Fréchet.

ii.) The pointwise multiplications

$$\cdot \colon \mathscr{C}^k(M) \times \mathscr{C}^k(M) \longrightarrow \mathscr{C}^k(M)$$

are continuous bilinear mappings.

iii.) The inclusions $\mathscr{C}^\ell \hookrightarrow \mathscr{C}^k$ are continuous and dense for all $0 \leq \ell \leq k$, where we define

$$\mathscr{C}^0(M) := \mathscr{C}(M).$$

iv.) The subspaces

$$\mathscr{C}^\ell_0(M) := \big\{ f \in \mathscr{C}^\ell(M) \colon \operatorname{supp} f \textit{ is compact} \big\} \subseteq \mathscr{C}^k(M)$$

of compactly supported $\mathscr{C}^\ell$-functions are dense for all $\ell = 0, \dots, k$.

Exercise 2.37 Prove Proposition 2.36.

Hint: For the first countability, one has to argue a bit that it is fine to exclude differential operators of order zero to obtain a countable defining system.

This leaves the question of how to topologize $\mathscr{C}^\ell_0(M)$ such that it becomes complete itself. If one is interested in distributions (in the sense of generalized functions), this turns out to be an unavoidable question. However, the appropriately conceptual answer is much more involved and ultimately uses final locally convex topologies. Unlike for initial topologies, one then has to demand to stay within the category of locally convex spaces. The interested reader can find the construction of the so-called inductive limit topology within [6, Sec. 4.5–4.6].

Before moving on, we construct the completion functor, which we have already teased a couple of times and will see frequent use in the remaining sections. To set the stage, let us precise what we want to understand by a completion.

Definition 2.38 (Completion) *A completion $(\widehat{V}, \iota)$ of a Hausdorff locally convex space V is a pair of a complete Hausdorff locally convex space $\widehat{V}$ and a linear embedding*

$$\iota\colon V \longrightarrow \widehat{V}$$

with dense image.

Exercise 2.39 Let V be a Hausdorff locally convex space with completion $(\widehat{V}, \iota)$. Prove the following:

i.) For every continuous linear mapping $L\colon V \longrightarrow W$ into another complete Hausdorff locally convex space there exists a unique continuous linear

$$\widehat{L}\colon \widehat{V} \longrightarrow W$$

such that $\widehat{L} \circ \iota = L$.

ii.) If (X, χ) is another completion of V, then there exists a unique linear homeomorphism

$$L\colon \widehat{V} \longrightarrow X \qquad \text{such that} \qquad L \circ \iota = \chi.$$

Example 2.40 Let V be a complete Hausdorff locally convex space. Then (V, id_V) constitutes a completion of V.

While our abstract approach makes the preceding statements simple to establish, it of course glosses over the issue that completions may not exist. To rectify this, we first observe that completion may be reduced to closure within some complete ambient space.

Lemma 2.41 *Let $\iota\colon V \longrightarrow W$ be a continuous linear embedding of a Hausdorff locally convex space V into a complete Hausdorff locally convex space W. Then the closure $\iota(V)^{\mathrm{cl}}$ together with the co-restriction*

$$\iota\colon V \longrightarrow \iota(V)^{\mathrm{cl}} \tag{2.16}$$

constitutes a completion of V.

Proof: Ultimately, there is not much left to be checked. As $\iota(V)^{\mathrm{cl}}$ is endowed with the (locally convex!) subspace topology inherited from W, (2.16) is still a continuous linear embedding. Moreover, its image $\iota(V)$ is dense within $\iota(V)^{\mathrm{cl}}$. □

Next, we return to Proposition 2.34 to discuss Cartesian products of locally convex spaces, as a sufficiently large product will provide the ambient space for completion.

Definition 2.42 *Let $(V_\alpha)_{\alpha \in J}$ be a collection of locally convex spaces. We define the Cartesian product topology on $\prod_{\alpha \in J} V_\alpha$ as the initial topology with respect to the canonical projections*

$$\pi_\beta\colon \prod_{\alpha \in J} V_\alpha \longrightarrow V_\beta$$

for all $\beta \in J$.

We collect the most important properties of locally convex Cartesian products.

Proposition 2.43 *Let $(V_\alpha)_{\alpha \in J}$ be a collection of locally convex spaces with corresponding defining systems of seminorms $(\mathcal{P}_\alpha)_{\alpha \in J}$ and Cartesian product $V := \prod_{\alpha \in J} V_\alpha$.*

i.) The Cartesian product topology is locally convex with defining system of seminorms

$$\mathcal{P} := \left\{ \sum_{\alpha \in F} \pi_\alpha^* \mathrm{q}_\alpha \colon F \subseteq J \textit{ finite}, \ \mathrm{q}_\alpha \in \mathcal{P}_\alpha \right\}. \tag{2.17}$$

ii.) The canonical inclusions

$$\iota_\beta \colon V_\beta \longrightarrow \prod_{\alpha \in J} V_\alpha$$

are linear embeddings with closed images for all $\alpha \in J$, i.e. ι_β is continuous and the subspace topology on $\iota_\beta(V_\beta)$ coincides with its original topology.

iii.) The Cartesian product topology on V is Hausdorff iff all of the V_α are.

iv.) The Cartesian product topology on V is complete iff all of the V_α are.

PROOF: The first part is just Proposition 2.34 again. We use the characteristic property of the initial topology to prove *ii.)*. Indeed, the continuity of ι_β is equivalent to the continuity of the compositions

$$\pi_\alpha \circ \iota_\beta = \begin{cases} \mathrm{id}_{V_\alpha} & \text{for } \alpha = \beta, \\ 0 & \text{for } \alpha \neq \beta, \end{cases}$$

for all $\alpha \in J$, which is obvious. Similarly, taking another look at (2.17), we get

$$\pi_\alpha^* \mathrm{q}_\alpha \Big|_{\iota_\beta(V_\beta)} = \begin{cases} \mathrm{q}_\alpha & \text{for } \alpha = \beta, \\ 0 & \text{for } \alpha \neq \beta \end{cases}$$

for all $\alpha, \beta \in J$ and $\mathrm{q}_\alpha \in \mathrm{cs}(V_\alpha)$. Hence, ι_β constitutes an embedding. The closedness of $\iota_\beta(V_\beta)$ within V is obvious. In view of Proposition 2.9, *vi.)*, part *iii.)*, is clear. The completeness is more interesting. Assume first the that V_α is complete Hausdorff for all $\alpha \in J$ and let $(v^k)_{k \in I} \subseteq V$ be a Cauchy net, where we use a Latin superscript as net index to distinguish it from the components of each element, which we denote by $v_\alpha^k \in V_\alpha$. By uniform continuity of the linear mappings π_β, see again Proposition 2.16, *i.)*, the component nets $(v_\alpha^k)_{k \in I} \subseteq V_\alpha$ are Cauchy for all $\alpha \in J$. By the assumed completeness and *iii.)*, they thus converge to unique limits $v_\alpha \in V_\alpha$ for all $\alpha \in J$. Let now $F \subseteq J$ be finite, $\mathrm{q}_\alpha \in \mathrm{cs}(V_\alpha)$ for all $\alpha \in F$ and $\varepsilon > 0$. By convergence of the components and finiteness of F, we find a corresponding index $k_0 \in I$ such that

$$\mathrm{q}_\alpha\big(v_\alpha^k - v_\alpha^k\big) \leq \varepsilon \qquad \text{for all } k \succcurlyeq k_0, \alpha \in F.$$

Let $v := (v_\alpha)_{\alpha \in J}$ be the componentwise limit. Then

$$\sum_{\alpha \in F} \pi_\alpha^* \mathrm{q}_\alpha\big(v^k - v\big) = \sum_{\alpha \in F} \mathrm{q}_\alpha\big(v_\alpha^k - v_\alpha^k\big) \leq |F| \cdot \varepsilon,$$

where $|F|$ denotes the number of elements of F. This proves $v^k \to v$, proving the completeness of V. Conversely, *ii.)* means that the V_α inherit the completeness of V. $\square$

Our considerations show that one may think of the Cartesian product topology as the topology of componentwise convergence. Working towards the completion functor, we are interested in a particular Cartesian product consisting of the so-called local Banach spaces. Thus, let us recall how to complete normed spaces $(N, \|\cdot\|)$, where we may use sequences instead of nets by virtue of Exercise 2.21. Consider the space

$$C := \Big\{(x_n)_n \in \prod_{n=0}^{\infty} N \colon (x_n)_n \text{ is Cauchy}\Big\},$$

of Cauchy sequences within N. We endow N with a locally convex topology based on the following observation.

Lemma 2.44 *Let N be a normed space. Then the limit*

$$\mathrm{q}\big((x_n)_n\big) := \lim_{n\to\infty} \|x_n\| \tag{2.18}$$

exists for all Cauchy sequences $(x_n)_n \subseteq N$ and defines a seminorm on C.

Proof: Ultimately, this boils down to the uniform continuity of

$$\|\cdot\| \colon N \longrightarrow \mathbb{R},$$

as such mappings preserve Cauchy sequences and $\mathbb{R}$ is complete. To establish the uniform continuity, recall that by the triangle inequality

$$\|x\| \le \|x-y\| + \|y\| \qquad \text{and} \qquad \|y\| \le \|x-y\| + \|x\|,$$

which implies

$$\big|\|x\| - \|y\|\big| \le \|x-y\|$$

for all $x, y \in N$. Hence, (2.18) is indeed well-defined and inherits non-negativity, homogeneity and the triangle inequality from the norm. That is to say, it constitutes the seminorm we seek. □

Note that (2.18) has a sizable kernel, namely the subspace of zero sequences $C_0 \subseteq C$. We endow C with the locally convex topology induced by q and its multiples. It turns out that passing to its Hausdorffization results in a completion of N.

Theorem 2.45 *Let N be a normed space. Then the pair $(C/C_0, \iota)$ with*

$$\iota \colon N \longrightarrow C/C_0, \quad \iota(x) := \big[(x)_n\big]$$

constitutes a completion of N. Here $(x)_n$ denotes the constant sequence with value $x \in N$. Moreover, the quotient C/C_0 is normable by

$$\big\|[(x_n)_n]\big\|_0 = \lim_{n\to\infty} \|x_n\| = \mathrm{q}\big((x_n)_n\big) \qquad \textit{for all } (x_n)_n \in C. \tag{2.19}$$

Proof: Observe first that the equivalence class $[(x)_n]$ consists of all sequences $(x_n)_n \subseteq N$ with $\lim_{n\to\infty} x_n = x$. This implies that ι constitutes an isometry, as

$$\mathrm{q}_0\big(\iota(x)\big) = \inf_{\lim x_n = x} \mathrm{q}\big((x_n)_n\big) = \inf_{\lim x_n = x} \lim_{n\to\infty} \|x_n\| = \|x\|$$

by continuity of the norm. In particular, ι is an embedding. Let now $(x_n)_n \in C$ be a Cauchy sequence within E and $\varepsilon > 0$. Then, by definition, there exists some $M \in \mathbb{N}$ with

$$\|x_n - x_m\| \le \varepsilon \qquad \text{for all } n, m \ge M.$$

In particular, this means $\|x_n - x_M\| \le \varepsilon$ for all $n \ge M$ and thus

$$\mathrm{q}\big((x_M)_n - (x_n)_n\big) = \lim_{n\to\infty} \|x_M - x_n\| \le \varepsilon.$$

That is to say, the constant sequence $(x_M)_n \in \mathrm{B}_\varepsilon((x_n)_n)$, proving the density of the constant sequences within C. By surjectivity and continuity of the quotient projection, this implies the density of $\iota(N)$ within C/C_0, see Exercise 2.47. Next, we verify (2.19). By (2.11), the locally convex topology of the quotient C/C_0 is generated by the multiples of q_0 with

$$\mathrm{q}_0\big([x]\big) = \inf_{c\in C_0} \mathrm{q}\big(x + c\big)$$

for all $x \in C$. Invoking the triangle inequality for q, we notice

$$\mathrm{q}(x) \le \mathrm{q}\big(x + c\big) + \mathrm{q}(c) = \mathrm{q}\big(x + c\big) \le \mathrm{q}(x) + \mathrm{q}(c) = \mathrm{q}(x)$$

for all $x \in C$ and $c \in C_0 = \ker \mathrm{q}_0$. Consequently,

$$\mathrm{q}_0\big([x]\big) = \inf_{c\in C_0} \mathrm{q}\big(x + c\big) = \inf_{c\in C_0} \mathrm{q}\big(x\big) = \mathrm{q}(x)$$

for all $x \in C$, which is (2.19). As C/C_0 is normed and thus first countable, we may use sequences to check its completeness by virtue of Exercise 2.21. Thus, let $([x^k])_k \subseteq C/C_0$ be a Cauchy sequence, where we once again put the sequence index as a superscript. Unwrapping the Cauchy condition for $\varepsilon > 0$ and using (2.19) yields some $K \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \|x_n^k - x_n^\ell\| = \mathrm{q}\big(x^k - x^\ell\big) = \mathrm{q}_0\big([x^k] - [x^\ell]\big) \le \varepsilon \qquad \text{for all } k, \ell \ge K. \tag{2.20}$$

Hence, for every $k, \ell \ge K$ there exists some $M(k, \ell) \ge \max\{k, \ell\}$ such that

$$\|x_n^k - x_n^\ell\| \le 2\varepsilon \qquad \text{for all } n \ge M(k, \ell).$$

Without loss of generality, we may choose $M(k, \ell)$ monotonically increasing with respect to each argument. Moreover, as each $x^k \in C$, there exist $M_k \in \mathbb{N}$ increasing with respect to k, such that $M_k \ge k$ and

$$\|x_n^k - x_m^k\| \le \frac{1}{k} \qquad \text{for all } n, m \ge M_k. \tag{2.21}$$

Consider now the sequence $y := (y_n)_n$ with entries $y_n := x_{M_n}^n$ for all $n \in \mathbb{N}$, which provides the candidate $[y]$ for the limit of $([x^k])$ within C/C_0. First, we check $y \in C$, i.e. have to verify the Cauchy condition. To this end, let $\varepsilon > 0$, $K \ge 1/\varepsilon$ as above and $n, m \ge K$. Then, setting $A := \max\{M_n, M_m, N(M_n, M_m)\}$, we may estimate

$$\begin{aligned}\|y_n - y_m\| &= \|x_{M_n}^n - x_{M_m}^m\| \\ &\le \|x_{M_n}^n - x_A^n\| + \|x_A^n - x_A^m\| + \|x_A^m - x_{M_m}^m\|\end{aligned}$$

$$\leq \frac{1}{n} + 2\varepsilon + \frac{1}{m}$$
$$\leq 4\varepsilon,$$

where we have used $A \geq N(M_n, M_m) \geq N(n, m)$ for the middle term. Note that, while the auxiliary index A depends on the concrete choice of n and m, the index K solely depends on ε. Hence, $y \in C$. It remains to establish the convergence of $[x^k]$ towards $[y]$. To this end, we have to estimate

$$\|y - x^k\|_0 = \lim_{n\to\infty} \|y_n - x_n^k\| = \lim_{n\to\infty} \|x_{M_n}^n - x_n^k\|$$

for sufficiently large $k \in \mathbb{N}$. Let again $\varepsilon > 0$. As $(y_n) \in C$, we find a corresponding $\alpha \in \mathbb{N}$ such that

$$\|x_{M_n}^n - x_{M_m}^m\| = \|y_n - y_m\| \leq \varepsilon \qquad \text{for all } n, m \geq \alpha.$$

Moreover, we may assume that $\alpha \geq 1/\varepsilon$ and $\alpha \geq K$ from (2.20). Combining this with (2.21) leads us to

$$\|y_n - x_n^k\| \leq \|y_n - y_\alpha\| + \|x_{M_\alpha}^\alpha - x_n^\alpha\| + \|x_n^\alpha - x_n^k\| \leq 2\varepsilon + \|x - x_n^k\|$$

for all $k \geq K$ and $n \geq M_\alpha$. Finally taking the limit $n \to \infty$ and invoking (2.20) thus gives

$$\|y - x^k\|_0 \leq 2\varepsilon + \lim_{n\to\infty} \|x_n^\alpha - x_n^k\| \leq 4\varepsilon$$

for all $k \geq K$. We have thus established $[x^k] \to y$ as desired. □

Remark 2.46 Our proof shows that completions $(\widehat{N}, \iota)$ of normed spaces N are normable. Moreover, one may achieve that the embedding $\iota\colon N \longrightarrow \widehat{N}$ constitutes an isometry. Indeed, this is the appropriate definition of completion within the category of normed spaces, whose morphisms consist of linear *contractions*. That is to say, linear mappings with operator norms bounded above by one. In this language, our observation may then be rephrased as "the completion as a normed space induces a completion of the underlying locally convex space". Following the naming scheme, we refer to the resulting functor as *Banachization*, which we have have for now only explained on objects and refers to the construction given within Theorem 2.45.

This is a opportune moment to clean up some considerations regarding density, which are also important for the action of the completion functor on morphisms.

Exercise 2.47 Let $f\colon X \longrightarrow Y$ be a continuous surjective mapping between topological spaces X and Y. Show that, if $D \subseteq X$ is dense, then so is $f(D) \subseteq Y$.

Exercise 2.48 Let V be a Hausdorff locally convex space and $D \subseteq V$ be a dense subspace. Moreover, let $L\colon D \longrightarrow W$ be a continuous linear mapping with values in a complete Hausdorff locally convex space. Show that there exists a unique continuous linear mapping

$$\widehat{L}\colon V \longrightarrow W \qquad \text{such that} \qquad \widehat{L}\Big|_D = L.$$

Hint: Think about uniqueness first, as it provides a natural candidate for $\widehat{L}$. Do not get distracted by the linearity, this is a topological statement.

Exercise 2.49 Let N be a normed space. Moreover, let $L\colon D \longrightarrow M$ be a continuous linear mapping defined on a dense subspace $D \subseteq N$ with values in a Banach space. Prove that its unique extension from Exercise 2.48 fulfils

$$\|\widehat{L}\| = \|L\|,$$

where $\|\cdot\|$ denotes the operator norm. Conclude that, if L is an isometry, so is $\widehat{L}$.

We are now in a position to define the local Banach spaces of a locally convex space V, which allow to reduce many locally convex questions back to normed ones.

Definition 2.50 (Local Banach Space) *Let V be a locally convex space. The local Banach space V_{q} at $\mathrm{q} \in \mathrm{cs}(V)$ is defined as the completion of the Hausdorffization of V endowed with the locally convex topology induced by* q *and its multiples.*

In symbols, this may be spelled out as

$$V_{\mathrm{q}} := \widehat{V/\ker \mathrm{q}}$$

with the Banachization functor $\widehat{\cdot}$. Summarizing, for every $\mathrm{q} \in \mathrm{cs}(V)$, the process of passing to the local Banach space V_{q} is given by

$$V \overset{\text{forget}}{\longrightarrow} (V, \mathrm{q}) \overset{\pi_{\mathrm{q}}}{\longrightarrow} (V, \|\cdot\|_{\mathrm{q}}) \overset{\iota_q}{\longrightarrow} (\widehat{V}, \|\cdot\|_{\mathrm{q}}),$$

where the first arrows forgets about all seminorms but q and all arrows are continuous linear mappings. Doing this for all $\mathrm{q} \in \mathrm{cs}(V)$ leads us now to the construction of a completion.

Theorem 2.51 *Let V be a Hausdorff locally convex space and endow the product*

$$\prod_{\mathrm{q} \in \mathrm{cs}(V)} V_{\mathrm{q}}$$

over the local Banach spaces $\mathrm{pr}_{\mathrm{q}}\colon V \longrightarrow V_{\mathrm{q}}$ with the Cartesian product topology. Then the pair $(\widehat{V}, \iota)$

$$\iota\colon V \longrightarrow \widehat{V}, \quad \big(\iota(v)\big)_{\mathrm{q}} := \mathrm{pr}_{\mathrm{q}}(v)$$

and $\widehat{V} := \iota(V)^{\mathrm{cl}}$ constitutes a completion of V.

Proof: As each local Banach space is complete Hausdorff, Proposition 2.43, *iii.)* and *iv.)* yield that $\widehat{V}$ is complete Hausdorff itself. Let now $v \in V \setminus \{0\}$. Invoking the Hausdorff property, see again Proposition 2.9, *vi.)*, we find a corresponding seminorm $\mathrm{q} \in \mathrm{cs}(V)$ such that $\mathrm{q}(v) > 0$. By (2.19), this means

$$\big\|\mathrm{pr}_{\mathrm{q}}(v)\big\|_{\mathrm{q}} = \mathrm{q}(v) > 0,$$

which in turn implies $\mathrm{pr}_{\mathrm{q}}(v) \neq 0$ as $\|\cdot\|_{\mathrm{q}}$ constitutes a norm. Hence, ι is injective. By the characteristic property of the initial topology, the continuity of ι follows from the continuity of its component mappings $\mathrm{pr}_{\mathrm{q}}\colon V \longrightarrow V_{\mathrm{q}}$ for all $\mathrm{q} \in \mathrm{cs}(V)$. Conversely, (2.19) implies that the subspace topology induced by the inclusion $\iota(V) \subseteq \widehat{V}$ is finer than the original topology. That is to say, ι constitutes an continuous linear embedding. Invoking Lemma 2.41 thus completes the proof. Pun very much intended. □

Combining the construction from Theorem 2.51 with Exercise 2.48 yields the desired functor at last.

Corollary 2.52 *The assignment*

$$\widehat{\cdot}\colon \mathsf{Lcs} \longrightarrow \mathsf{Lcs}$$

constitutes a covariant functor.

It is instructive to check that Banachization has the same pleasant properties.

Exercise 2.53 Let $L\colon N \longrightarrow M$ be a linear contraction between normed spaces with Banachizations $(\widehat{N}, \iota_N)$ and $(\widehat{M}, \iota_M)$. Prove that there exists a unique linear contraction

$$\widehat{L}\colon \widehat{N} \longrightarrow \widehat{M} \qquad \text{such that} \qquad \widehat{L} \circ \iota_V = \iota_W \circ L.$$

We conclude the section with trying out our new machinery within a simple example. To facilitate this, we first observe that we may replace the index set $\operatorname{cs}(V)$ for the ambient Cartesian product within Theorem 2.51 by a defining system $\mathcal{P}$ of seminorms for V. Even more convenient, this system needs not be filtrating either.

Example 2.54 Let J be an index set and consider the space of finite sequences $c_{00}(J)$ from (2.7), endowed with the topology of pointwise convergence. As the diligent reader has already worked out, we get a defining system of seminorms by taking finite sums of

$$\mathrm{q}_\alpha := |a_\alpha|,$$

where we vary $\alpha \in J$ through the elements of J. The corresponding local Banach space is thus given by the projection

$$\mathrm{pr}_\alpha\colon c_{00}(J) \longrightarrow \mathbb{C}, \quad (\mathrm{pr}_F\, a)_\alpha := a_\alpha$$

onto the components. Hence, the ambient space is simply

$$\prod_{\alpha \in J} \mathbb{C} \cong \operatorname{Map}(J, \mathbb{C})$$

of all complex sequences endowed with the topology of pointwise convergence and $\iota(c_{00}(J))$ reproduces $c_{00}(J)$. In view of Exercise 2.14, its closure is all of $\operatorname{Map}(J, \mathbb{C})$ and we have thus found a completion of $c_{00}(J)$. It is instructive to go through the construction again, but for finite sums of the seminorms q_α as a defining system instead. Then the ambient space is strictly bigger than the completion, as $\iota(c_{00}(J))$ then corresponds to consistent sequences only.

3 Projective and Injective Tensor Products

There is a plethora of well written treatments of projective tensor products such as [8, §41] and [6, Sec. 15]. The typical discussions of injective tensor products suffer from a distinct lack of focus on seminorms, and as such the lecture notes [15, Sec. 6.3] are the most recommendable source for further reading. We begin our considerations with the central example we would like to understand.

Example 3.1 We consider the Banach space $\mathscr{C}([0,1])$ of complex-valued continuous functions on the unit interval. Its tensor square

$$V := \mathscr{C}([0,1]) \otimes \mathscr{C}([0,1])$$

naturally embeds into $\mathscr{C}([0,1] \times [0,1])$ by means of linearly extending

$$\big(f \otimes g\big)(x, y) := f(x) \cdot g(y). \tag{3.1}$$

The resulting space constitutes a point separating *-subalgebra of $\mathscr{C}([0,1] \times [0,1])$ containing the constant functions and as such is dense with respect to the topology of uniform convergence on the unit square $[0,1] \times [0,1]$ by virtue of the Stone-Weierstraß Theorem.

The goal of this section is, roughly speaking, to endow the algebraic tensor product $V \otimes W$ of two locally convex spaces V and W with a locally convex topology derived solely from the topologies of the constituent spaces. Remarkably, one has a multitude of reasonable choices here, a problem famously explored by Grothendieck's [3] during his time as a PhD student. In these notes, we restrict ourselves to injective and projective tensor products, both of which enjoy a plethora of pleasant properties and may be described by explicit defining system of seminorms. Moreover, they pave the way for a conceptually pleasing definition of nuclearity.

We begin with the injective flavour. The principal observation is that for any element $\varphi \in V'$ of the (continuous) dual space of V, its absolute value $|\varphi|$ defines a continuous seminorm on V. Such seminorms are particularly simple, as they come with a sizable kernel.

Definition 3.2 *Let V and W be locally convex spaces.*

i.) For $\mathrm{q} \in \mathrm{cs}(V)$ and $\mathrm{p} \in \mathrm{cs}(W)$, we define their injective tensor product as

$$\big(\mathrm{q} \otimes_\varepsilon \mathrm{p}\big)(x) := \sup\Big\{ \big|(v' \otimes w')(x)\big| : v' \in V', |v'| \le \mathrm{q}, w' \in W', |w'| \le \mathrm{p} \Big\} \tag{3.2}$$

for all $v \in V$ and $w \in W$.

ii.) The injective tensor product topology on $V \otimes W$ is generated by the seminorms $\mathrm{q} \otimes_\varepsilon \mathrm{p}$, where we vary $\mathrm{q} \in \mathrm{cs}(V)$ and $\mathrm{p} \in \mathrm{cs}(W)$. We write $V \otimes_\varepsilon W$ for the resulting locally convex space.

The linearity of v' and w' ensures that (3.2) indeed constitutes a seminorm. It is straightforward to verify that it suffices to take injective tensor products of seminorms from defining systems of V and W. Moreover, the resulting system is indeed filtrating.

Remark 3.3 (Polars) Let V and W be locally convex spaces. The polar of a subset $A \subseteq V$ is defined by

$$A^{\circledast} := \big\{ v' \in V' : \big|v'(v)\big| \le 1 \text{ for all } v \in A \big\}.$$

The condition on v' and w' within (3.2) may thus be rephrased as

$$v' \in \mathrm{B}_{\mathrm{q}}^{\circledast} := (\mathrm{B}_{\mathrm{q},1}(0)^{\mathrm{cl}})^{\circledast} \subseteq V' \qquad \text{and} \qquad w' \in \mathrm{B}_{\mathrm{p}}^{\circledast} := (\mathrm{B}_{\mathrm{p},1}(0)^{\mathrm{cl}})^{\circledast} \subseteq W',$$

respectively. As the closed cylinders constitute zero neighbourhoods, the Banach-Alaoglu Theorem asserts that these polars are compact with respect to the weak*-topology, i.e. the topology of pointwise convergence. If V is normed, then

$$\mathrm{B}^{\circledast}_{\|\cdot\|} = \left\{v' \in V' \colon \forall_{v \in V} \; \left|v'(v)\right| \le \|v\|\right\} = \mathrm{B}_1(0)^{\mathrm{cl}},$$

where we endow V' with the operator norm. This recovers the weak* compactness of the unit ball in V'. Thus polars may be regarded as the locally convex incarnation of these balls. More comprehensive discussions can be found within [].

Exercise 3.4 Let V be normed. Interpret the statement of Exercise 2.23 in the context of its bidual V''.

The following reformulation of (3.2) is often useful, as it only uses one supremum. The idea is that we may pair away a factor of the tensor product to obtain a linear mapping from the continuous dual space to the remaining factors.

Lemma 3.5 *Let V and W be locally convex spaces. Then*

$$\begin{aligned} \iota_V \colon V \otimes W &\longrightarrow \mathrm{L}(V', W), \quad \iota_V(v \otimes w)v' := v'(v) \cdot w \\ \iota_W \colon V \otimes W &\longrightarrow \mathrm{L}(W', V), \quad \iota_W(v \otimes w)w' := w'(w) \cdot v \end{aligned}$$

extend to well-defined linear injections. Moreover,

$$\big(\mathrm{q} \otimes_\varepsilon \mathrm{p}\big)(x) = \sup_{|v'| \le \mathrm{q}} \mathrm{p}\big(\iota_V(x)v'\big) = \sup_{|w'| \le \mathrm{p}} \mathrm{q}\big(\iota_W(x)w'\big) \tag{3.3}$$

for all $x \in V \otimes W$, $\mathrm{q} \in \mathrm{cs}(V)$ and $\mathrm{p} \in \mathrm{cs}(W)$.

Note that we have to endow V' and W' with locally convex topologies to speak of $\mathrm{L}(V', W)$ and $\mathrm{L}(W', V)$. As we shall show in a moment, any reasonable choice works, as the mappings are already continuous with respect to the topology of pointwise convergence.

Proof: The continuity of $\iota_V(v \otimes w)$ follows from the estimate

$$\mathrm{p}\big(\iota_V(x)v'\big) \le \sum_k \mathrm{p}\big(\iota_V(v_k \otimes w_k)v'\big) = \sum_k \left|v'(v_k)\right| \cdot \mathrm{p}(w_k)$$

for all $v' \in V'$, $\mathrm{p} \in \mathrm{cs}(W)$ and $x = \sum_k v_k \otimes w_k \in V \otimes W$ at once. Indeed, fixing x and setting

$$\mathrm{r}_v \colon V' \longrightarrow \mathbb{R}, \quad \mathrm{r}_v(v') := \left|v'(v)\right|,$$

we have shown the continuity estimate

$$\mathrm{p}\big(\iota_V(x)v'\big) \le \sum_k \mathrm{r}_{v_k}(v') \cdot \mathrm{p}(w_k)$$

for all $v' \in V'$ and $\mathrm{p} \in \mathrm{cs}(W)$. In particular, $\iota_V(x)$ is continuous for all finer topologies on V'. The continuity of $\iota_W(x)$ follows by swapping the roles of V and W. Unwrapping (3.3) and using Exercise 2.23, we get

$$\sup_{|v'| \le \mathrm{q}} \mathrm{p}\big(\iota_V(x)v'\big) = \sup_{|v'| \le \mathrm{q}} \mathrm{p}\left(\sum_k v'(v_k) \cdot w_k\right)$$

$$
\begin{aligned}
&= \sup_{|v'|\leq \mathrm{q}} \sup_{|w'|\leq \mathrm{p}} \left| w'\Big(\sum_k v'(v_k)\cdot w_k\Big)\right| \\
&= \sup_{|v'|\leq \mathrm{q}} \sup_{|w'|\leq \mathrm{p}} \left| \sum_k v'(v_k)\cdot w'(w_k)\right| \\
&= \big(\mathrm{q} \otimes_\varepsilon \mathrm{p}\big)(x).
\end{aligned}
$$

The other equality follows analogously. ☐

Using this, we may show that $\otimes_\varepsilon$ formalizes vector-valued versions of many familiar spaces.

Example 3.6 Let M be a set and consider the normed space

$$
\mathscr{B}(M) \coloneqq \Big\{ f\colon M \longrightarrow \mathbb{C} \colon \|f\|_\infty \coloneqq \sup_{x\in M} \big|f(x)\big| < \infty \Big\}
$$

of all globally bounded complex-valued mappings defined on M. Moreover, let V be locally convex. We are interested in the tensor product $\mathscr{B}(M)\otimes V$, which turns out to be related to vector-valued globally bounded functions on M. In this context, we call a function $F\colon M \longrightarrow V$ globally bounded if $F(M) \subseteq V$ is bounded, i.e. $\sup_{z\in M} \mathrm{q}(F(z)) < \infty$ for all $\mathrm{q} \in \mathrm{cs}(V)$. Hence, the space $\mathscr{B}(M,V)$ carries a natural locally convex topology induced by the seminorms

$$
\mathrm{q}_M(F) \coloneqq \sup_{z\in M} \mathrm{q}\big(F(z)\big),
$$

where we vary $\mathrm{q} \in \mathrm{cs}(V)$. Notably, we may embed $\mathscr{B}(M)\otimes V$ into $\mathscr{B}(M,V)$ by linearly extending

$$
\big(f \otimes v\big)(z) \coloneqq f(z)\cdot v. \tag{3.4}
$$

The tensor product $\mathscr{B}(M)\otimes V$ then corresponds precisely of the mappings with images contained in finite dimensional bounded subsets of V. Using (3.3) and (2.9), we compute

$$
\begin{aligned}
\big(\|\cdot\|_\infty \otimes_\varepsilon \mathrm{q}\big)(x) &= \sup_{|v'|\leq \mathrm{q}} \big\|\iota_V(x)v'\big\|_\infty \\
&= \sup_{|v'|\leq \mathrm{q}} \sup_{z\in M} \big|\iota_V(x)v'\big| \\
&= \sup_{z\in M} \sup_{|v'|\leq \mathrm{q}} \left|\sum_k v'(v_k)\cdot f_k(z)\right| \\
&= \sup_{z\in M} \sup_{|v'|\leq \mathrm{q}} \left|v'\Big(\sum_k v_k\cdot f_k(z)\Big)\right| \\
&= \sup_{z\in M} \mathrm{q}\big(x(z)\big) \\
&= \mathrm{q}_M(x)
\end{aligned}
$$

for $\mathrm{q} \in \mathrm{cs}(V)$ and $x = \sum_k f_k \otimes v_k \in \mathscr{B}(M)\otimes V$. Hence, (3.4) extends to a linear embedding

$$
\mathscr{B}(M)\otimes_\varepsilon V \hookrightarrow \mathscr{B}(M,V). \tag{3.5}
$$

If V is complete, this embedding has dense range, so that

$$
\mathscr{B}(M,V) \cong \mathscr{B}(M)\widehat{\otimes}_\varepsilon V. \tag{3.6}
$$

We leave the remaining details as an exercise.

Exercise 3.7 Let M be a set and V a complete Hausdorff locally convex space.

i.) Prove that $\mathscr{B}(M, V)$ is complete Hausdorff.

ii.) Show that (3.5) has dense range.

iii.) Conclude (3.6).

Instead, we return to Example 3.1.

Lemma 3.8 *Using the identification* (3.1)*, we have*

$$\mathscr{C}([0,1]) \mathbin{\widehat{\otimes}_\varepsilon} \mathscr{C}([0,1]) \cong \mathscr{C}\big([0,1] \times [0,1]\big)$$

as locally convex spaces.

Proof: It suffices to prove that the injective tensor product topology translates to the topology of uniform convergence on $[0,1] \times [0,1]$ through the identification (3.1). Let

$$F = \sum_k f_k \otimes g_k \in \mathscr{C}([0,1]) \otimes \mathscr{C}([0,1]).$$

As $\mathscr{C}([0,1]) \subseteq \mathscr{B}([0,1])$ is a subspace, we may repeat our computation from Example 3.6 to obtain

$$\big(\|\cdot\|_\infty \otimes_\varepsilon \|\cdot\|_\infty\big)(F) = \sup_{x \in [0,1]} \big\|F(x, \cdot)\big\|_\infty = \sup_{x,y \in [0,1]} \big|F(x,y)\big|.$$

Thus (3.1) is an embedding. We have already argued within Example 3.1 that its image is dense by virtue of the Stone-Weierstraß Theorem. $\square$

Exercise 3.9 *i.)* Generalize Lemma 3.8 to $\mathscr{C}(U)$ for open sets $U \subseteq \mathbb{R}^n$.

ii.) Extend the statement further to smooth functions $\mathscr{C}^\infty(U)$ for open sets $U \subseteq \mathbb{R}^n$.

iii.) Globalize the assertion to $\mathscr{C}^\infty(U)$ for open sets $U \subseteq M$ of a smooth manifold M.

It is convenient to study the abstract properties of injective and projective tensor products in tandem, so we proceed with yet another definition.

Definition 3.10 (Projective Tensor Products) *Let V and W be locally convex spaces.*

i.) For $\mathrm{q} \in \mathrm{cs}(V)$ and $\mathrm{p} \in \mathrm{cs}(W)$, we define their projective tensor product as

$$\big(\mathrm{q} \otimes_\pi \mathrm{p}\big)(x) := \inf\Big\{\sum_k \mathrm{q}(v_k) \cdot \mathrm{p}(w_k) \colon x = \sum_k v_k \otimes w_k\Big\}. \tag{3.7}$$

ii.) The projective tensor product topology on $V \otimes W$ is generated by the seminorms $\mathrm{q} \otimes_\pi \mathrm{p}$, where we vary $\mathrm{q} \in \mathrm{cs}(V)$ and $\mathrm{p} \in \mathrm{cs}(W)$. We write $V \otimes_\pi W$ for the resulting locally convex space.

As the possible decompositions of a linear combination $x + \lambda y \in V \otimes W$ are in bijection with the decompositions of x and y individually, we get that (3.7) indeed constitute seminorms.

Exercise 3.11 Let V and W be locally convex spaces and $\mathrm{q} \in \mathrm{cs}(V)$ and $\mathrm{p} \in \mathrm{cs}(W)$. Prove that (3.2) and (3.7) constitute seminorms on $V \otimes W$.

Again, it suffices to use defining systems of seminorms and the resulting system is filtrating. The main reason to study injective and projective tensor products simultaneously is that the latter is always larger than the former.

Lemma 3.12 ($\otimes_\pi$ vs. $\otimes_\varepsilon$, [14, Cor. of Prop. 43.4]) *Let V and W be locally convex spaces. Then the identity mapping*

$$V \otimes_\pi W \longrightarrow V \otimes_\varepsilon W$$

is continuous. More precisely, if $\mathrm{p} \in \mathrm{cs}(V)$ and $\mathrm{q} \in \mathrm{cs}(W)$, then

$$\mathrm{p} \otimes_\varepsilon \mathrm{q} \leq \mathrm{p} \otimes_\pi \mathrm{q}. \tag{3.8}$$

Proof: Let $x \in V \otimes W$ and consider a finite decomposition

$$x = \sum_k v_k \otimes w_k$$

into factorizing tensors as well as $v' \in \mathrm{B}_\mathrm{p}^{*}$ and $w' \in \mathrm{B}_\mathrm{q}^{*}$. Then

$$\left|\left(v' \otimes w'\right)(x)\right| \leq \sum_k \left|v'(v_k) \cdot w'(w_k)\right| \leq \sum_k \mathrm{p}(v_k) \cdot \mathrm{q}(w_k).$$

Note that the left-hand side is independent of the chosen decomposition, whereas the right-hand side does not depend on the choice of v' and w'. Thus, taking the infimum over all decompositions of x and the supremum over all v' and w' in the respective polars proves the continuity estimate (3.8). $\square$

Due to the supremum within (3.2) and infimum in (3.7), it is typically rather cumbersome to compute projective and injective tensor products explicitly. The triangle inequality combined with the following Lemma provides a simple yet powerful estimate from above.

Lemma 3.13 *Let V and W be locally convex spaces. Then*

$$\left(\mathrm{p} \otimes_\varepsilon \mathrm{q}\right)(v \otimes w) = \mathrm{p}(v) \cdot \mathrm{q}(w) = \left(\mathrm{p} \otimes_\pi \mathrm{q}\right)(v \otimes w) \tag{3.9}$$

for all $\mathrm{p} \in \mathrm{cs}(V)$, $\mathrm{q} \in \mathrm{cs}(W)$, $v \in V$ and $w \in W$.

That is to say, projective and injective tensor products factorize on factorizing tensors. The proof is based on the Hahn-Banach Theorem and showcases the interplay between both concepts.

Proof (of Lemma 3.13): Let $\mathrm{p} \in \mathrm{cs}(V)$, $\mathrm{q} \in \mathrm{cs}(W)$, $v \in V$ and $w \in W$. By (3.7), we have

$$\left(\mathrm{p} \otimes_\pi \mathrm{q}\right)(v \otimes w) \leq \mathrm{p}(v) \cdot \mathrm{q}(w)$$

as $v \otimes w$ already comes with a decomposition into factorizing tensors out of the box. For the converse inequality, we consider the linear functionals

$$v' \colon \operatorname{span} v \longrightarrow \mathbb{C}, \quad v'(\lambda \cdot v) := \lambda \cdot \mathrm{p}(v)$$

$$w' \colon \operatorname{span} w \longrightarrow \mathbb{C}, \quad w'(\lambda \cdot w) \coloneqq \lambda \cdot \mathrm{q}(w),$$

which fulfil $|v'| \le \mathrm{p}$ and $|w'| \le \mathrm{q}$ by construction. Invoking the Hahn-Banach Theorem **??**, we may thus extend both continuously to all of V and W' while preserving the continuity estimates. That is to say, $v' \in V'$ and $w' \in W'$ get to partake in the supremum within (3.2), which yields the estimate

$$\big(\mathrm{p} \otimes_\varepsilon \mathrm{q}\big)(v \otimes w) \ge \Big| \big(v' \otimes w'\big)(v \otimes w) \Big| = \big| v'(v) \cdot w'(w) \big| = \mathrm{p}(v) \cdot \mathrm{q}(w).$$

Putting everything together and using (3.8), we arrive at

$$\mathrm{p}(v) \cdot \mathrm{q}(w) \ge \big(\mathrm{p} \otimes_\pi \mathrm{q}\big)(v \otimes w) \ge \big(\mathrm{p} \otimes_\varepsilon \mathrm{q}\big)(v \otimes w) \ge \mathrm{p}(v) \cdot \mathrm{q}(w),$$

which is (3.9). □

Viewing (3.9) as a continuity estimate, the following is immediate.

Corollary 3.14 *Let V and W be locally convex spaces. Then*

$$\otimes_\varepsilon \colon V \times W \longrightarrow V \otimes_\varepsilon W \qquad \textit{and} \qquad \otimes_\pi \colon V \times W \longrightarrow V \otimes_\pi W$$

are continuous bilinear mappings.

Another natural question is whether tensor products of norms are again norms. In view of the infimum within (3.7), this is not completely obvious if done directly. That being said, (3.8) ensures that it suffices to provide a lower bound for the injective tensor product (3.2). Here, the supremum in the definition comes to our aid much like in the proof of Lemma 3.13. This is a common theme. More generally, this question may be recast as the inheritance of the Hausdorff property upon taking tensor products. The precise statement is the following.

Proposition 3.15 (Hausdorff Property) *Let V and W be locally convex spaces.*

- *i.) If $\mathrm{q} \in \mathrm{cs}(V)$ and $\mathrm{p} \in \mathrm{cs}(W)$ are norms, then so are $\mathrm{q} \otimes_\varepsilon \mathrm{p}$ and $\mathrm{q} \otimes_\pi \mathrm{p}$.*
- *ii.) The tensor products $V \otimes_\varepsilon W$ and $V \otimes_\pi W$ are Hausdorff if V and W are. The converse holds whenever $V \neq \{0\} \neq W$.*

PROOF: We begin with the second statement and assume first that V and W are Hausdorff. Let $x = \sum_{k=1}^N v_k \otimes w_k \in V \otimes W$. Without loss of generality, we may assume

$$v_1 \notin \operatorname{span}\{v_2, \ldots, v_N\};$$

otherwise we write v_1 as a linear combination of the remaining vectors and simplify back to multiples of $v_2, \ldots, v_N$. Relabeling and repeating the process then terminates after finitely many steps, achieving the desired condition. By assumption, we find $\mathrm{q} \in \mathrm{cs}(V)$ with $\mathrm{q}(v_1) > 0$ and $\mathrm{p} \in \mathrm{cs}(W)$ with $\mathrm{p}(w_1) > 0$. Similar to before, we consider the linear functionals

$$v' \colon \operatorname{span}\{v_1, \ldots, v_N\} \longrightarrow \mathbb{C}, \quad v'\Bigg(\sum_{k=1}^N \lambda_k v_k\Bigg) \coloneqq \lambda \cdot \mathrm{q}(v_1)$$

$$w' \colon \operatorname{span} w_1 \longrightarrow \mathbb{C}, \quad w'(\lambda \cdot w_1) \coloneqq \lambda \cdot \mathrm{p}(w_1).$$

Again, $|v'| \leq \mathrm{q}$ and $|w'| \leq \mathrm{p}$, and we use the Hahn-Banach Theorem **??** to continuously extend both to V and W, respectively, while preserving the continuity estimates. Together with (3.8), we arrive at

$$\big(\mathrm{q} \otimes_\pi \mathrm{p}\big)(x) \geq \big(\mathrm{q} \otimes_\varepsilon \mathrm{p}\big)(x) \geq \big|(v' \otimes w')(x)\big| = \left|\sum_{k=1}^N \underbrace{v'(v_k)}_{=\delta_{k,1} \cdot \mathrm{q}(v_1)} w'(w_k)\right| = \mathrm{q}(v_1) \cdot \mathrm{p}(w_1) > 0.$$

Varying x establishes the Hausdorff property of $V \otimes_\pi W$ and $V \otimes_\varepsilon W$. Assume conversely that neither V, nor W are the zero space and $V \otimes_\pi W$ or $V \otimes_\varepsilon W$ is Hausdorff. By choosing some $v \in V \setminus \{0\}$, we get a linear injection

$$\iota_v \colon W \longrightarrow V \otimes W, \quad \iota_v(w) \coloneqq v \otimes w.$$

By virtue of Lemma 3.13, it even constitutes a topological embedding for either topology. That is to say, it is a homeomorphism onto its image. Hence, W inherits the Hausdorff property from the ambient space. The argument for V is analogous. This completes the proof of *ii.)* and we turn towards *i.)*. To this end, assume that $\mathrm{q} \in \mathrm{cs}(V)$ and $\mathrm{p} \in \mathrm{cs}(W)$ are norms. Then we may regard (V, q) and (W, p) as normed spaces, whose locally convex topology is generated by

$$\big\{r \cdot \mathrm{q} \colon r > 0\big\} \qquad \text{resp.} \qquad \big\{r \cdot \mathrm{p} \colon r > 0\big\}.$$

As tensor products of Hausdorff spaces are Hausdorff by *ii.)*, we get that the defining system

$$\big\{(r_1 \cdot \mathrm{q}) \otimes_{\pi/\varepsilon} (r_2 \cdot \mathrm{p}) \colon r_1, r_2 > 0\big\} = \big\{r \cdot (\mathrm{q} \otimes_{\pi/\varepsilon} \mathrm{p}) \colon r > 0\big\}$$

has no joint kernel. But this means

$$\mathrm{q} \otimes_{\pi/\varepsilon} \mathrm{p}$$

is a norm itself, completing the proof. □

It is instructive to make the argument we have just used precise.

Exercise 3.16 Let $(V, \|\cdot\|)$ be a normed space. Prove that the system of seminorms

$$\big\{r \cdot \|\cdot\| \colon r > 0\big\}$$

is filtrating and that the associated locally convex topology reproduces the norm-topology.

Another pleasant property of both flavours of tensor products is that naively gluing continuous linear mappings preserves continuity. We begin with the injective version.

Lemma 3.17 *Let $\phi_j \colon V_j \longrightarrow W_j$ be continuous linear mappings between locally convex spaces for $j = 1, 2$. Then the mapping*

$$\phi_1 \otimes \phi_2 \colon V_1 \otimes_\varepsilon V_2 \longrightarrow W_1 \otimes_\varepsilon W_2$$

defined by linear extension of

$$\big(\phi_1 \otimes \phi_2\big)(v_1 \otimes v_2) \coloneqq \phi_1(v_1) \otimes \phi_2(v_2)$$

is continuous.

PROOF: Let $\mathrm{p}_1 \in \mathrm{cs}(W_1)$ and $\mathrm{p}_2 \in \mathrm{cs}(W_2)$. By continuity of ϕ_1 and ϕ_2, there exist corresponding seminorms $\mathrm{q}_1 \in \mathrm{cs}(V_1)$ and $\mathrm{q}_2 \in \mathrm{cs}(V_2)$ such that

$$\phi_1^* \mathrm{p}_1 = \mathrm{p}_1 \circ \phi_1 \leq \mathrm{q}_1 \qquad \text{and} \qquad \phi_2^* \mathrm{p}_2 = \mathrm{p}_2 \circ \phi_2 \leq \mathrm{q}_2.$$

Consequently, we have

$$\mathrm{B}^{\circledast}_{\phi^* \mathrm{p}_1} \subseteq \mathrm{B}^{\circledast}_{\mathrm{q}_1} \qquad \text{and} \qquad \mathrm{B}^{\circledast}_{\phi^* \mathrm{p}_2} \subseteq \mathrm{B}^{\circledast}_{\mathrm{q}_2}.$$

Writing $\phi \coloneqq \phi_1 \otimes \phi_2$, this implies

$$\begin{aligned}
\big(\mathrm{p}_1 \otimes_\varepsilon \mathrm{p}_2\big)(\phi(x)) &= \sup_{w_1' \in \mathrm{B}^{\circledast}_{\mathrm{p}_1}} \sup_{w_2' \in \mathrm{B}^{\circledast}_{\mathrm{p}_2}} \Big| \big(w_1' \otimes w_2'\big)(\phi(x)) \Big| \\
&= \sup_{w_1' \in \mathrm{B}^{\circledast}_{\mathrm{p}_1}} \sup_{w_2' \in \mathrm{B}^{\circledast}_{\mathrm{p}_2}} \Big| \big(\phi_1^* w_1' \otimes \phi_2^* w_2'\big)(x) \Big| \\
&\leq \sup_{v_1' \in \mathrm{B}^{\circledast}_{\mathrm{q}_1}} \sup_{v_2' \in \mathrm{B}^{\circledast}_{\mathrm{q}_2}} \big| \big(v_1' \otimes v_2'\big)(x) \big| \\
&\leq \big(\mathrm{q}_1 \otimes_\varepsilon \mathrm{q}_2\big)(x),
\end{aligned}$$

which completes the proof. □

Exercise 3.18 Prove that taking injective and projective tensor products is associative up to the usual linear algebraic identifications. That is to say, show that

$$\mathrm{p}_1 \otimes_{\pi/\varepsilon} \big(\mathrm{p}_2 \otimes_{\pi/\varepsilon} \mathrm{p}_3\big) = \big(\mathrm{p}_1 \otimes_{\pi/\varepsilon} \mathrm{p}_2\big) \otimes_{\pi/\varepsilon} \mathrm{p}_3$$

as mappings on the triple tensor product $V_1 \otimes V_2 \otimes V_3$ of locally convex spaces V_1, V_2, V_3 and $\mathrm{p}_j \in \mathrm{cs}(V_j)$ for $j = 1, 2, 3$. Conclude that the resulting locally convex spaces are canonically linearly homeomorphic.

Recall the universal property of the algebraic tensor product: Given a bilinear mapping

$$\phi \colon V \times W \longrightarrow X,$$

there exists a unique linear mapping

$$\Phi \colon V \otimes W \longrightarrow X \quad \text{such that } \Phi \circ \otimes = \phi.$$

Our Corollary 3.14 ensures that the continuity of Φ implies the continuity of ϕ. The natural question is thus whether the converse also holds. This turns out to be a distinguishing property of the projective tensor product, which we could have used as the definition instead. This statement is known as the *infimum argument* and an ubiquitous tool within strict deformation quantization as discussed in the survey [16]. Using the associativity from Exercise 3.18, the general statement is the following.

Proposition 3.19 (Infimum argument, [14, Prop. 43.4]) *Let $V_1, \ldots, V_n, W$ be locally convex spaces and*

$$\phi \colon V_1 \times \cdots \times V_n \longrightarrow W$$

be n-linear with corresponding linear map

$$\Phi \colon V_1 \otimes \cdots \otimes V_n \longrightarrow W.$$

Endow $V_1\times\cdots\times V_n$ with the Cartesian product topology and $V_1\otimes\cdots\otimes V_n$ with the projective tensor product topology. Then ϕ is continuous if and only if Φ is. More precisely, if for a continuous seminorm $\mathrm{q}\in\mathrm{cs}(W)$ there are $\mathrm{p}_1\in\mathrm{cs}(V_1)$, …, $\mathrm{p_n}\in\mathrm{cs}(V_n)$ such that

$$\mathrm{q}\big(\phi(v_1,\ldots,v_n)\big)\le\mathrm{p}_1(v_1)\cdots\mathrm{p}_n(v_n)\qquad\text{for all } v_1\in V_1,\ldots,v_n\in V_n, \tag{3.10}$$

then

$$\mathrm{q}\big(\Phi(v)\big)\le\big(\mathrm{p}_1\otimes\cdots\otimes\mathrm{p}_n\big)(v)\qquad\text{for all } v\in V_1\otimes\cdots\otimes V_n, \tag{3.11}$$

and vice versa.

Proof: It suffices to treat the case $n=2$. As already noted, we may view Lemma 3.13 as the continuity estimate

$$\big(\mathrm{p}_1\otimes_\pi\mathrm{p}_2\big)(v_1\otimes v_2)\le\mathrm{p}_1(v_1)\cdot\mathrm{p}_2(v_2).$$

Hence, (3.10) implies (3.11) by virtue of $\Phi\circ\otimes_\pi=\phi$. Assume conversely (3.11) and let

$$x=\sum_k v_k\otimes w_k\in V_1\otimes V_2.$$

Then we have

$$\mathrm{q}\big(\Phi(x)\big)\le\sum_k\mathrm{q}\big(\Phi(v_k\otimes w_k)\big)=\sum_k\mathrm{q}\big(\phi(v_k,w_k)\big)\le\sum_k\mathrm{p}_1(v_k)\cdot\mathrm{p}_2(w_k).$$

Taking the infimum over all decompositions of x into factorizing tensors thus yields

$$\mathrm{q}\big(\Phi(x)\big)\le\big(\mathrm{p}_1\otimes_\pi\mathrm{p}_2\big)(x).$$

Varying $x\in V_1\otimes V_2$ thus establishes (3.11). □

The upshot is that it suffices to prove continuity estimates for multilinear mappings on factorizing tensors. As a simple application, we may prove the analogue of Lemma 3.17 for projective tensor products.

Corollary 3.20 *Let $\phi_j\colon V_j\longrightarrow W_j$ be continuous linear mappings between locally convex spaces for $j=1,2$. Then the mapping*

$$\phi_1\otimes\phi_2\colon V_1\otimes_\pi V_2\longrightarrow W_1\otimes_\pi W_2$$

defined by linear extension of

$$\big(\phi_1\otimes\phi_2\big)(v_1\otimes v_2)\coloneqq\phi_1(v_1)\otimes\phi_2(v_2)$$

is continuous.

Proof: By Proposition 3.19 and Corollary 3.14, it suffices to prove the continuity of the bilinear mapping

$$\phi\colon V_1\times V_2\longrightarrow W_1\times W_2,\quad \phi(v_1,v_2)\coloneqq\big(\phi(v_1),\phi(v_2)\big).$$

By continuity of ϕ_1 and ϕ_2 and the definition of the product topology, this is obvious. □

Example 3.21 Let V be a complete Hausdorff locally convex space. A sequence of vectors $\gamma \coloneqq (v_n) \subseteq V$ is called absolutely summable if

$$\mathrm{q}(\gamma) \coloneqq \sum_{n=0}^{\infty} \mathrm{q}(v_n) < \infty \tag{3.12}$$

for all $\mathrm{q} \in \mathrm{cs}(V)$. We write $\ell^1[V]$ for the corresponding locally convex space. The usual ℓ^1-space then arises for $V = (\mathbb{C}, |\cdot|)$, viewed as a locally convex space. We claim that

$$\ell^1 \widehat{\otimes}_\pi V \cong \ell^1[V] \tag{3.13}$$

as locally convex spaces. To see this, we consider the mapping

$$\phi \colon \ell^1 \widehat{\otimes}_\pi V \longrightarrow \ell^1[V]$$

defined by linear extension of

$$\big(\phi(a \otimes v)\big)_n \coloneqq a_n \cdot v.$$

By virtue of

$$\mathrm{q}\big(\phi(a \otimes v)\big) = \sum_{n=0}^{\infty} \mathrm{q}\big(a_n \cdot v\big) = \sum_{n=0}^{\infty} |a_n| \cdot \mathrm{q}(v) = \|a\|_1 \cdot \mathrm{q}(v) \tag{3.14}$$

for all $a \in \ell^1$ and $v \in V$ and Proposition 3.19, this is indeed well-defined on all of the completion. Moreover, the resulting mapping is clearly injective on $\ell^1 \otimes_\pi V$ and thus the same is true for its unique continuous linear extension. Let now $\gamma \in \ell^1[V]$. We check that the series

$$\Gamma \coloneqq \sum_{n=0}^{\infty} e_n \otimes \gamma_n$$

with the usual unit sequences $e_n \in \ell^1$ given by $e_n(k) \coloneqq \delta_{n,k}$ converges within $\ell^1 \widehat{\otimes}_\pi V$. Indeed, by Lemma 3.13 we have

$$\sum_{n=0}^{\infty} \big(\|\cdot\|_1 \otimes_\pi \mathrm{q}\big)(e_n \otimes \gamma_n) = \sum_{n=0}^{\infty} \|e_n\|_1 \cdot \mathrm{q}\big(\gamma_n\big) = \sum_{n=0}^{\infty} \mathrm{q}\big(\gamma_n\big) = \mathrm{q}(\gamma) < \infty$$

for all $\mathrm{q} \in \mathrm{cs}(V)$. By completeness, we know that absolute convergence implies convergence, and thus $\Gamma \in \ell^1 \widehat{\otimes}_\pi V$ is well-defined. By continuity of ϕ and the completeness of V, this implies the convergence of

$$\phi(\Gamma) = \sum_{n=0}^{\infty} \phi\big(e_n \otimes \gamma_n\big) = \sum_{n=0}^{\infty} e_n \cdot \gamma_n = \gamma.$$

Hence, ϕ is also surjective and thus bijective. Finally, reading (3.14) backwards asserts the continuity of the inverse mapping. Note that passing to the completion was essential. The image

$$\phi\big(\ell^1 \otimes_\pi V\big)$$

only consists of sequences living within finite dimensional subspaces of V. As a particular consequence of the equality (3.14) combined with Proposition 3.19, we get that

$$\ell^1 \widehat{\otimes}_\pi \ell^1[J] \cong \ell^1[\mathbb{N}_0 \times J]$$

for any index set J. More generally, one may use the same methods to prove

$$\ell^1[I] \widehat{\otimes}_\pi \ell^1[J] \cong \ell^1[I \times J].$$

Finally, we note that there exist measure theoretic generalizations, which can be found within [6, Sec. 16.7]. It should be noted that this particularly nice compatibility is particular to ℓ^1 and fails e.g. for the space of zero sequences, see [5, Ex. 2.6.2] for further discussion. Another such example is the Hilbert space ℓ^2, where one should use the Hilbert tensor product instead.

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4 Nuclearity

The de-facto source regarding nuclearity remains [12], even though some of the treatment has become dated. A more modern point of view can be found in the lecture notes [15].

In Lemma 3.12, we have seen that there is a canonical continuous linear mapping

$$V \otimes_\pi W \longrightarrow V \otimes_\varepsilon W$$

between the projective and injective tensor products of two locally convex spaces V and W. It turns out that requiring this mapping to even constitute a homeomorphism has far reaching consequences. This is the notion of nuclearity.

Definition 4.1 (Nuclearity) *We call a locally convex space V nuclear if the canonical mapping*

$$V \otimes_\pi W \longrightarrow V \otimes_\varepsilon W$$

constitutes a homeomorphism for all locally convex spaces W.

In the sequel, we may thus suppress the index for the tensor product, whenever we work with nuclear spaces.

From the Definition, it is clear that we need some better criteria to check nuclearity.

Proposition 4.2 *... maybe sketch some examples?*

There are many more equivalent, yet different looking and thus situationally useful characterizations of nuclearity. Our next goal is to generalize the Riemann rearrangement Theorem to nuclear spaces. We have already met the space of absolutely summable sequences $\ell^1[V]$ of a complete Hausdorff space within Example 3.21. We managed to identify it as the completion of the projective tensor product of ℓ^1 and V within (3.13). We prove the analogous result holds for injective tensor products and unconditionally convergent series.

Example 4.3 (Uncoditional Summability I) Let V be a complete Hausdorff locally convex space and $(v_n)_n \subseteq V$ a sequence. Recall that the corresponding series

$$\sum_{n=0}^{\infty} v_n$$

is called unconditionally convergent to $v \in V$ if

$$\sum_{n=0}^{\infty} v_{\sigma(n)} = v$$

for all bijections $\sigma\colon \mathbb{N}_0 \longrightarrow \mathbb{N}_0$. That is to say, one may rearrange the series without spoiling convergence or altering the limit. We write $\ell^1(V)$ for the resulting vector space. At this point, it is not completely clear how one should topologize $\ell^1(V)$. Note first that the canonical mapping

$$\phi\colon \ell^1 \otimes V \longrightarrow \ell^1(V), \quad \big(\phi(\gamma \otimes v)\big)_n \coloneqq \gamma_n \cdot v \tag{4.1}$$

constitutes a linear injection. Indeed, that this sequence is indeed unconditionally summable by the usual Riemann Rearrangement Theorem, as every tensor is a finite linear combination of factorizing tensors, and thus the resulting limit is taken within a finite dimensional subspace of V. Invoking once again (3.3), we compute

$$\begin{aligned}
\big(\|\cdot\|_1 \otimes_\varepsilon \mathrm{q}\big)(x) &= \sup_{|v'|\le \mathrm{q}} \big\|\iota_V(x)v'\big\|_1 \\
&= \sup_{|v'|\le \mathrm{q}} \bigg\|\sum_k v'(v_k)\cdot \gamma_k\bigg\|_1 \\
&= \sup_{|v'|\le \mathrm{q}} \sum_{n=0}^{\infty} \bigg|\sum_k v'(v_k)\cdot\gamma_k(n)\bigg| \\
&= \sup_{|v'|\le \mathrm{q}} \sum_{n=0}^{\infty} \bigg|v'\bigg(\sum_k v\cdot\gamma_k(n)\bigg)\bigg| \\
&= \sup_{|v'|\le \mathrm{q}} \sum_{n=0}^{\infty} \big|v'(x_n)\big|
\end{aligned}$$

for $\mathrm{q} \in \mathrm{cs}(V)$ and $x = \sum_k \gamma_k \otimes v_k \in \ell^1 \otimes V$, where we write

$$x_n \coloneqq \sum_k (\gamma_k)(n)\cdot v_k \in V \qquad \text{for all } n \in \mathbb{N}_0.$$

Hence, a reasonable choice of seminorms turns out to be

$$\mathrm{q}_w\big((v_n)_n\big) \coloneqq \sup_{|v'|\le \mathrm{q}} \sum_{n=0}^{\infty} \big|v'(v_n)\big|, \tag{4.2}$$

where we vary $\mathrm{q} \in \mathrm{cs}(V)$. The index w alludes to weak convergence. It is not completely obvious that the suprema within (4.2) are finite. To see this, we take a detour to note the following characterization of unconditional convergence.

Lemma 4.4 *Let V be a Hausdorff locally convex space and $(v_n)_n \subseteq V$ a sequence. Then the series $\sum_{n \in \mathbb{N}_0} v_n$ converges unconditionally to $v \in V$ iff for every $\varepsilon > 0$ and $\mathrm{q} \in \mathrm{cs}(V)$ there exists a finite set $F_0 \subseteq \mathbb{N}_0$ such that*

$$\mathrm{q}\Bigg(v - \sum_{n \in F} v_n\Bigg) \leq \varepsilon \tag{4.3}$$

for all finite sets $F_0 \subseteq F \subseteq \mathbb{N}_0$.

PROOF: Assume first that the condition from the Lemma holds and let $\sigma\colon \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ be a bijection. Moreover, let $\varepsilon > 0$ and $\mathrm{q} \in \mathrm{cs}(V)$. By assumption, we find some finite set $F_0 \subseteq \mathbb{N}_0$ such that (4.3) holds for all finite sets $F_0 \subseteq F \subseteq \mathbb{N}_0$. As F_0 is finite, there exists some $N_0 \in \mathbb{N}_0$ such that $\sigma(n) \geq \max F_0$ for all $n \geq N$. That is to say by bijectivity of σ, we have

$$F_0 \subseteq \sigma\big(\{1, \ldots, N_0\}\big),$$

which implies

$$\mathrm{q}\Bigg(v - \sum_{n=0}^{N} v_{\sigma(n)}\Bigg) \leq \varepsilon$$

for all $N \geq N_0$. Varying ε and q establishes the convergence of $\sum_{n=0}^{\infty} v_{\sigma(n)}$ towards v.

Assume, conversely, that the condition stated in the Lemma fails. Then there exists some constant $\varepsilon > 0$ and $\mathrm{q} \in \mathrm{cs}(V)$ such that for every finite set $F_0 \subseteq \mathbb{N}_0$, there exists another finite set $F_0 \subseteq F \subseteq \mathbb{N}_0$ such that

$$\mathrm{q}\Bigg(v - \sum_{n \in F} v_n\Bigg) > \varepsilon.$$

We use this to define a bijection $\sigma\colon \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ as follows: first, we set $\sigma(0) := 0$. If now $\sigma(0), \ldots, \sigma(n)$ are already defined for some $n \in \mathbb{N}_0$, then take $F_0 := \{\sigma(0), \ldots, \sigma(n)\}$ with corresponding finite set $F_0 \subseteq F \subseteq \mathbb{N}_0$. If $F = F_0$, then we set

$$\sigma(n+1) := \min\big(\mathbb{N}_0 \setminus F_0\big).$$

Otherwise, we define

$$\sigma(n+1), \sigma(n+2), \ldots, \sigma(n + |F \setminus F_0|)$$

as the elements of $F \setminus F_0$ in ascending order. Note that the finite sets F within each step contain all the preceding ones. Inductively, this defines a bijection σ. We check that the accordingly rearranged series does not converge to v. To this end, let $N_0 \in \mathbb{N}_0$. Then, by construction of σ, there exists some $N_1 \geq N_0$ such that

$$F(N_0) = \big\{\sigma(0), \ldots, \sigma(N_1)\big\}.$$

This implies the desired inequality

$$\mathrm{q}\Bigg(v - \sum_{n=0}^{N_1} v_{\sigma(n)}\Bigg) > \varepsilon.$$

Example 4.5 (Uncoditional Summability II) Let V be a complete Hausdorff locally convex space. We return to the finiteness of (4.2) for unconditionally summable sequences $(v_n)_n \subseteq V$ with limit v. Fix $v' \in V$. Invoking Lemma 4.4, we find some finite set $F_0 \subseteq \mathbb{N}_0$ such that

$$\left| v'(v) - \sum_{n \in F} v'(v_n) \right| = \left| v'\Big(v - \sum_{n \in F} v_n\Big) \right| \leq 1$$

for all finite sets $F_0 \subseteq F \subseteq \mathbb{N}_0$, where we use that $|v'| \in \mathrm{cs}(V)$ by continuity of v'. Consequently,

$$\sum_{n \in J} \big| v'(v_n) \big| \leq 1 + \big| v'(v) \big|$$

for all finite sets $F_0 \subseteq F \subseteq \mathbb{N}_0$. Hence, the series

$$\sum_{n=0}^{\infty} v'(v_n)$$

converges absolutely within $\mathbb{C}$, as the corresponding sequence of partial sums is bounded from above and monotonically increasing. Hence, each individual term within the supremum (4.2) is finite. The remainder of the proof uses technology beyond the scope of these notes. For the sake of completeness, we give the argument regardless. Consider the sets

$$\Phi_r := \Big\{ v' \in V' \colon \sum_{n=0}^{\infty} \big| v'(v_n) \big| \leq r \Big\} = \bigcap_{J \in \mathcal{F}} \Big\{ v' \in V' \colon \sum_{n \in J} \big| v'(v_n) \big| \leq r \Big\} \tag{4.4}$$

for all $r > 0$ and where $\mathcal{F}$ denotes the collection of all finite subsets of $\mathbb{N}_0$. By what we have just shown, we know

$$\bigcup_{r>0} \Phi_r = \bigcup_{r>0} r \cdot \Phi_1 = V'.$$

Moreover, Φ_r is absolutely convex and closed with respect to the topology of pointwise convergence on V'. Indeed, the closedness of each of the members of the intersection in (4.4) follows from the finiteness of J and thus Φ_r is closed as an intersection of closed sets. We have shown that each Φ_r, in particular Φ_1, constitutes a *barrel* within V'. Hence, pulling back the intersection $\Phi_1 \cap \mathrm{B}^{\circledast}_{\mathrm{q}}$ to functionals on the local Banach space

$$V_{\mathrm{q}} := \widehat{V/\ker \mathrm{q}}$$

with the obvious quotient norm $[\mathrm{q}]$ yields a barrel $[\Phi] \subseteq V'_{\mathrm{q}}$. As a Banach space, V_{q} is barrelled and thus $[\Phi]$ constitutes a zero neighbourhood and is thus contained in some sufficiently large norm-ball. That is to say, there exists $r > 0$ such that

$$\sum_{n=0}^{\infty} \big| v'(v_n) \big| = \sum_{n=0}^{\infty} \big| [v]'([v_n]) \big| \leq r \qquad \text{for all } v' \in \mathrm{B}^{\circledast}_{\mathrm{q}},$$

where we have used that $\ker \mathrm{q} \subseteq \ker v'$. Hence, (4.2) is indeed well-defined. In passing, we note that our argument did not rely on the unconditional convergence and may be readily generalized to arbitrary summable sequences. This gives rise to yet another complete space of sequences on V, which constitutes yet another contender for which space the symbol $\ell^1(V)$ should really denote. Within these notes, it shall unnamed and thus shrouded in mystery. Our next goal is to show

$$\ell^1 \widehat{\otimes}_\varepsilon V \cong \ell^1(V), \tag{4.5}$$

where we once again use the assumed completeness of V. As in the projective situation, we establish the density of the inclusion (4.1). Let $(v_n)_n \in \ell^1(V)$ with $v := \sum_{n=0}^\infty v_n$ and consider the sequence (γ_m) given by

$$\gamma_m := \sum_{n=0}^m e_n \otimes v_n \in \ell^1 \otimes_\varepsilon V.$$

Given $\varepsilon > 0$ and q, we invoke Lemma 4.4 to find a finite set $F_0 \subseteq \mathbb{N}_0$ such that

$$\mathrm{q}\Big(v - \sum_{n \in F} v_n\Big) \leq \varepsilon$$

for all finite sets $F_0 \subseteq F \subseteq \mathbb{N}_0$. Obversely, this means that

$$\mathrm{q}\Big(\sum_{n \in F} v_n\Big) \leq \mathrm{q}\Big(v - \sum_{n \in F_0} v_n\Big) + \mathrm{q}\Big(v - \sum_{n \in F \cup F_0} v_n\Big) \leq 2\varepsilon$$

for any finite set $F \subseteq \mathbb{N}_0$ with $F \cap F_0 = \emptyset$. In particular, we get

$$\Big| v'\Big(\sum_{n \in F} v_n\Big)\Big| = \Big|\sum_{n \in F} v'(v_n)\Big| \leq 2\varepsilon$$

for any finite set $F \subseteq \mathbb{N}_0$ with $F \cap F_0 = \emptyset$ and $v' \in V'$ with $|v'| \leq \mathrm{q}$. Covering $\mathbb{N}_0$ with the sets

$$\mathbb{N}_\pm^\pm := \big\{n \in \mathbb{N}_0 \colon \pm \operatorname{Re} v'(v_n) \geq 0, \pm \operatorname{Im} v'(v_n) \geq 0\big\}$$

leads to the estimate

$$\begin{aligned}
\sum_{n \in F} \big|v'(v_n)\big| &\leq \sum_{n \in F} \big|\operatorname{Re} v'(v_n)\big| + \sum_{n \in F} \big|\operatorname{Im} v'(v_n)\big| \\
&= \sum_{n \in F \cap \mathbb{N}_+^+} \big|\operatorname{Re} v'(v_n)\big| + \sum_{n \in F \cap \mathbb{N}_-^+} \big|\operatorname{Re} v'(v_n)\big| \\
&\quad + \sum_{n \in F \cap \mathbb{N}_+^+} \big|\operatorname{Im} v'(v_n)\big| + \sum_{n \in F \cap \mathbb{N}_+^-} \big|\operatorname{Im} v'(v_n)\big| \\
&\leq 8\varepsilon
\end{aligned}$$

for the same data and where we have used the triangle inequality in the plane to infer

$$|z| = |\operatorname{Re} z + \operatorname{Im} z| \leq |\operatorname{Re} z| + |\operatorname{Im} z| \qquad \text{for all } z \in \mathbb{C}.$$

Varying the data, we get

$$\sup_{|v|' \leq \mathrm{q}} \sum_{n > N} \big|v'(v_n)\big| \leq 8\varepsilon.$$

By construction, we have on the other hand

$$v_n - \gamma_m(n) = \begin{cases} 0 & \text{for } n = 0, \dots, m, \\ v_n & \text{for } n > m, \end{cases}$$

and thus

$$\mathrm{q}_w\big((v_n)_n - \gamma_m\big) = \sup_{|v'| \leq \mathrm{q}} \sum_{n=0}^{\infty} \Big| v'\big(v_n - \gamma_m(n)\big)\Big| = \sup_{|v'| \leq \mathrm{q}} \sum_{n=m+1}^{\infty} \big|v'(v_n)\big| \leq 8\varepsilon$$

for all $m \geq N$. This proves the desired convergence $\gamma_m \to (v_n)_n$.

Once again, it remains to establish the completeness of the ambient space to complete the argument.

Exercise 4.6 Let V be a complete Hausdorff locally convex space. Prove that the space $\ell^1(V)$ of unconditionally summable sequences is complete Hausdorff.

In passing, we note the elementary estimate

$$\mathrm{q}_w\big((v_n)_n\big) = \sup_{|v'| \leq \mathrm{q}} \sum_{n=0}^{\infty} \big|v'(v_n)\big| \leq \sum_{n=0}^{\infty} \mathrm{q}(v_n) = \mathrm{q}\big((v_n)_n\big) \qquad \text{for all } (v_n)_n \in \ell^1[V],$$

with the seminorms for $\ell^1[V]$ from (3.12). In view of (3.13) and (4.5) this is nothing but a particular case of (3.8). We may encode this as a canonical linear injection

$$\ell^1[V] \longrightarrow \ell^1(V).$$

Putting everything together, we arrive at the following Theorem.

Theorem 4.7 (Summability in Nuclear spaces) *Let V be a complete Hausdorff nuclear locally convex space. Then $\ell^1[V] \cong \ell^1(V)$. That is to say, any series within V converges absolutely iff it converges unconditionally.*

PROOF: Combining the very definition of nuclearity with (3.13) and (4.5) yields

$$\ell^1[V] \cong \ell^1 \widehat{\otimes}_\pi V \cong \ell^1 \widehat{\otimes}_\varepsilon V \cong \ell^1(V).$$

The additional statement follows from unwrapping the canonical homeomorphisms. □

This constitutes a direct generalization of Riemann's Rearrangement Theorem beyond the finite-dimensional situation. Note that the completeness is immaterial and may always be remedied by passing to the completion of V as a preliminary step. Remarkably, the equality of unconditional and absolute summability characterizes nuclearity by [?]. However, this means that infinite-dimensional normed spaces are never nuclear.

Example 4.8 Consider $V := \ell^2$ and write e_n for the usual orthogonal basis with entries given by $e_n(k) = \delta_{n,k}$ for all $n, k \in \mathbb{N}$. Then the series

$$\sum_{n=1}^{\infty} \frac{e_n}{n}$$

converges unconditionally, but not absolutely. Hence, ℓ^2 is not nuclear by Theorem 4.7. By virtue of the Riesz-Fischer Theorem, this extends to any infinite dimensional Hilbert space. In fact, the same is true for any infinite dimensional *normed* space by an application of the highly non-trivial Dvoretzky-Rogers Theorem [2].

We proceed with pursuing another remarkable property of nuclear spaces: the existence of compact sets with non-empty open interior. That is to say, we are going to prove a variant of Montel's Theorem for nuclear spaces. In passing, we introduce quasicompleteness, which is a weaker notion of completeness useful in many applications. Recall that every Cauchy sequence is bounded. The same need not be true about Cauchy nets. It is instructive to play with this within some examples.

Exercise 4.9 Construct a unbounded Cauchy net. Argue that your net converges within the completion of your space, yet remains unbounded there.

This leads to the insight that unbounded Cauchy nets might not be the ones we care about. Accordingly, there is a more appropriate flavour of completeness.

Definition 4.10 (Quasicompleteness) *A locally convex space V is called quasicomplete if every* bounded *Cauchy net converges.*

Theorem 4.11 *Let V be a nuclear as well as quasicomplete space. Then every bounded subset $B \subseteq V$ has compact closure.*

Proof: This is an elaborate exercise for another day. □

5 Incomplete Discussion of a certain Example

The goal of this Section is to study a rather instructive example of a conveniently smooth, but discontinuous quadratic polynomial on $\mathscr{C}_c^\infty(\mathbb{R})$, the space of test functions on the real line. Our protagonist is given by

$$P\colon \mathscr{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}, \quad P(\phi) \coloneqq \sum_{n=0}^{\infty} \phi(n) \cdot \phi^{(n)}(0), \tag{5.1}$$

where $\phi^{(n)}$ denotes the n-th derivative of ϕ. By compactness of its support, the sum within (5.1) is only formally infinite. Moreover, P is a quadratic polynomial induced by the symmetric bilinear mapping

$$\check{P}(\phi, \psi) \coloneqq \sum_{n=0}^{\infty} \phi(n) \cdot \psi^{(n)}(0).$$

As we did not delve into the definition of strict inductive limits such as $\mathscr{C}_c^\infty(\mathbb{R})$ in the remainder of the text, we simply collect its properties we shall need without further explanation or proof. Its topology is the final locally convex (!) topology with respect to the inclusions of

$$\mathscr{C}_K^\infty(\mathbb{R}) \coloneqq \big\{\phi \in \mathscr{C}^\infty(\mathbb{R})\colon \operatorname{supp}\phi \subseteq K\big\}$$

for compact subsets $K \subseteq \mathbb{R}$, all of which we endow with the subspace topology inherited from $\mathscr{C}^\infty(\mathbb{R})$. By continuity of the evaluation functionals, each of the $\mathscr{C}_K^\infty(\mathbb{R})$ is closed within $\mathscr{C}^\infty(\mathbb{R})$ and as such a Fréchet space itself. Note moreover that

$$\mathscr{C}_c^\infty(\mathbb{R}) = \bigcup_{n=1}^{\infty} \mathscr{C}_{[-n,n]}^\infty(\mathbb{R}),$$

so it suffices to consider a countable and ascending collection of Fréchet spaces.

Proposition 5.1 (Properties of $\mathscr{C}_c^\infty(\mathbb{R})$)

i.) *The canonical inclusions $\mathscr{C}_K^\infty(\mathbb{R}) \hookrightarrow \mathscr{C}_c^\infty(\mathbb{R})$ are topological embeddings, i.e. continuous linear mappings such that the subspace topology induced by the inclusion reproduces the original topology.*

ii.) *A seminorm p on $\mathscr{C}_c^\infty(\mathbb{R})$ is continuous iff its restrictions fulfil*

$$\mathrm{p}\Big|_{\mathscr{C}_K^\infty(\mathbb{R})} \in \mathrm{cs}\big(\mathscr{C}_K^\infty(\mathbb{R})\big)$$

for all compact subsets $K \subseteq \mathbb{R}$.

iii.) *Let $B \subseteq \mathscr{C}_c^\infty(\mathbb{R})$ be bounded. Then there exists a compact set $K \subseteq \mathbb{R}$ such that $B \subseteq \mathscr{C}_K^\infty(\mathbb{R})$ and B is bounded there.*

iv.) *Let $(\phi_n)_n \subseteq \mathscr{C}_c^\infty(\mathbb{R})$ be a convergent sequence with limit ϕ. Then there exists some $N \in \mathbb{N}_0$ such that $\phi, \phi_n \in \mathscr{C}_{[-N,N]}^\infty(\mathbb{R})$ and $(\phi_n)_n$ converges to ϕ within this space.*

PROOF: We only note that *iv.)* follows directly by combining *i.)* with *iii.)*, as convergent sequences are always bounded. □

The following is a convenient concrete description of a defining system of seminorms for $\mathscr{C}_c^\infty(\mathbb{R})$. The idea is that by 5.1, *ii.)*, on every compact set, a continuous seminorm may differentiate only a finite number of times, while the global rank may be unbounded. Instead of exhausting $\mathbb{R}$ by compact subsets, we now use the good cover given by

$$K_0 := [-2, 2] \qquad \text{and} \qquad K_n := [-(n+2), -n] \cup [n, n+2] \qquad \text{for all } n \in \mathbb{N}.$$

This is to minimize overlaps while still covering $\mathbb{R}$ with the open interiors. Given a sequence $a := (a_n)$ of non-negative real numbers and $d := (d_n) \subseteq \mathbb{N}_0$, we define the seminorm

$$\mathrm{p}_{a,d}(\phi) := \sum_{n=0}^{\infty} a_n \cdot \max_{x \in K_n} \max_{0 \le k \le d_n} \big|\phi^{(k)}(x)\big|.$$

As within (5.1), the series is only formally infinite. Moreover, their restrictions to each of the spaces $\mathscr{C}_{K_n}^\infty(\mathbb{R})$ generate its locally convex topology for all $n \in \mathbb{N}_0$.

Lemma 5.2 *The collection*

$$\mathcal{P} := \big\{\mathrm{p}_{a,d} \colon a, d \in \mathbb{R}^{\mathbb{N}},\, \forall_{n \in \mathbb{N}_0} \colon a_n \ge 0 \wedge d_n \in \mathbb{N}_0\big\}$$

constitutes a defining system of seminorms for $\mathscr{C}_c^\infty(\mathbb{R})$.

PROOF: Note first that $\mathcal{P}$ is filtrating, as taking the pointwise maxima of both parametrizing sequences generates a joint upper bound at once. The continuity of each $\mathrm{p}_{a,d}$ is clear by Proposition 5.1, *ii.)*. Conversely, let $\mathrm{p} \in \mathrm{cs}(\mathscr{C}_c^\infty(\mathbb{R}))$. Then, once again invoking Proposition 5.1, *ii.)*, we know that the restrictions

$$\mathrm{p}\Big|_{\mathscr{C}_{K_n}^\infty(\mathbb{R})}$$

are continuous for all $n \in \mathbb{N}_0$. Hence, by our preliminary observations, we find $a_n > 0$ and $d_n \in \mathbb{N}_0$ such that

$$\mathrm{p}(\phi) \le \max_{x \in K_n} \max_{0 \le k \le d_n} \big|\phi^{(k)}(x)\big| \qquad \text{for all } \phi \in \mathscr{C}_{K_n}^\infty(\mathbb{R}).$$

Repeating this for all $n \in \mathbb{N}_0$ yields sequences a and d with components a_n and d_n for all $n \in \mathbb{N}_0$. Let now $(\chi_n)_{n\in\mathbb{N}_0}$ be a smooth partition of unity subordinate to the cover consisting of the sets

$$U_n := K_n^\circ \qquad \text{for all } n \in \mathbb{N}_0.$$

That is to say, we demand $\operatorname{supp}\chi_n \subseteq U_n^{\mathrm{cl}} = K_n$ for all $n \in \mathbb{N}_0$. Then, our construction yields

$$\mathrm{p}(\phi) \leq \sum_{n=0}^{\infty} \mathrm{p}\big(\underbrace{\chi_n \cdot \phi}_{\in \mathscr{C}^\infty_{K_n}(\mathbb{R})}\big) \leq \sum_{n=0}^{\infty} \max_{x\in K_n} \max_{0\leq k\leq d_n} \big|\phi^{(k)}(x)\big| = \mathrm{p}_{a,d}(\phi)$$

for all $\phi \in \mathscr{C}^\infty_c(\mathbb{R})$. This completes the proof. □

In passing, we note that this technique may be readily generalized to the test functions on any smooth manifold M. The non-trivial topological ingredients are then a good covering and a subordinate partition of unity.

Theorem 5.3 *The polynomial P from* (5.1) *is sequentially continuous, bounded, conveniently smooth but discontinuous.*

PROOF: The first three statements are based on the observation that the restrictions

$$P\Big|_{\mathscr{C}^\infty_{[-N,N]}(\mathbb{R})} \colon \mathscr{C}^\infty_{[-N,N]}(\mathbb{R}) \longrightarrow \mathscr{C}^\infty_{[-N,N]}(\mathbb{R})$$

are continuous for all $N \in \mathbb{N}$. Indeed, we have the continuity estimates

$$\big|P(\phi)\big| = \left|\sum_{n=0}^{N} \phi(n) \cdot \phi^{(n)}(0)\right| \leq (N+1) \cdot \sup_{x\in[-N,N]} \big|\phi(x)\big| \cdot \sup_{\ell=0,\dots,n} \big|\phi^{(\ell)}(0)\big|$$

for all $\phi \in \mathscr{C}^\infty_{[-N,N]}(\mathbb{R})$ and $N \in \mathbb{N}_0$.

Let now $(\phi_n)_n \subseteq \mathscr{C}^\infty_c(\mathbb{R})$ be a convergent sequence with limit $\phi \in \mathscr{C}^\infty_c(\mathbb{R})$. By Proposition 5.1, *iv.)* there exists some $N \in \mathbb{N}$ such that $\phi, \phi_n \in \mathscr{C}^\infty_{[-N,N]}(\mathbb{R})$ for all $n \in \mathbb{N}_0$ and (ϕ_n) converges to ϕ within this space. By our preliminary consideration, this means that $P(\phi_n) \to P(\phi)$, which is what we have wanted to establish.

The boundedness follows a similar line of reasoning. Indeed, if $B \subseteq \mathscr{C}^\infty_c(\mathbb{R})$ is bounded, then there exists some $N \in \mathbb{N}_0$ with $B \subseteq \mathscr{C}^\infty_{[-N,N]}(\mathbb{R})$ and B is bounded there by Proposition 5.1, *iii.)*. Using again our preliminary consideration then implies the boundedness of $P(B)$, as continuous polynomials are bounded by [5, Lemma 2.1.9 *ii.)*].

For the convenient smoothness, consider a continuous curve $\gamma\colon \mathbb{R} \longrightarrow \mathscr{C}^\infty_c(\mathbb{R})$ and let $x_0 \in \mathbb{R}$. By continuity, the trace

$$B := \gamma\big([x_0 - 1, x_0 + 1]\big) \subseteq \mathscr{C}^\infty_c(\mathbb{R})$$

is compact and thus bounded. Invoking again Proposition 5.1, *iii.)*, there thus exists a compact set $K \subseteq \mathbb{R}$ such that $B \subseteq \mathscr{C}^\infty_K(\mathbb{R})$. But then the composition

$$P\Big|_K \circ \gamma\Big|_{[x_0-1,x_0+1]} \colon [x_0 - 1, x_0 + 1] \longrightarrow \mathbb{C}$$

is well-defined. By our preliminary consideration, the restriction $P\big|_K$ is a continuous polynomial and thus Fréchet holomorphic. This implies Bastiani smoothness by [5, Rem. 2.3.19] and as $\mathscr{C}^\infty_K(\mathbb{R})$ is Fréchet, also convienent smoothness. Alternatively, one views $P\big|_K$ as a finite sum of products of linear functionals, whose smoothness may be readily verified directly.

Finally, we turn towards the discontinuity of P. By what we have already established, we have to find a zero net $(\phi_\alpha)_\alpha$ such that $P(\phi_\alpha) \not\to 0$. To this end, we use the closed unit cylinders corresponding to the seminorms $\mathrm{p}_{a,d}$ from Lemma 5.2 as directed set. To this end, let $a = (a_n)_n, d = (d_n)_n$ be real sequences with $a_n > 0$ and $d_n \in \mathbb{N}_0$ for all $n \in \mathbb{N}_0$. Our goal is now to construct a corresponding $\phi_{a,d} \in \mathscr{C}^\infty_c(\mathbb{R})$ such that $\mathrm{p}_{a,d}(\phi_{a,d}) = 1 = P(\phi_{a,d})$, as then $\phi_{a,d} \to 0$. Indeed, this follows immediately from the obvious monotonicity of our seminorms. Our first ingredient are functions $\phi_n \in \mathscr{C}^\infty_{[-1/2,1/2]}(\mathbb{R})$ such that

$$\max_{x\in K}\max_{0\le k\le d_n} \big|\phi_n^{(k)}(x)\big| = \frac{1}{a_n} = \phi_n(n).$$

The construction of these functions is the content of Exercise 5.5. For $n = 0$ we additionally demand that $\phi \equiv 1/a_0$ on $[-1/4, 1/4]$. This has the pleasant consequence

$$\phi_0^{(k)}(0) = 0 \qquad \text{for all } k \ge 1.$$

The other ingredient is simply the exponential function and another sequence

$$(\delta_n)_n \in \{0,1\}^{\mathbb{N}} \qquad \text{with } \delta_n = 0 \text{ for almost all } n \in \mathbb{N}.$$

Consider now

$$\phi_{a,d,\delta}(x) := \phi_0(x)\cdot \exp(2\kappa\cdot x) + \sum_{n=1}^\infty \frac{\delta_n}{2^n}\cdot \phi_n(x+n),$$

where $\kappa > 0$ is a parameter and the δ_n ensure that the sum is finite and $\operatorname{supp}\phi_{a,d,\delta}$ is compact. By construction, the supports of different terms do not overlap, so we have

$$\mathrm{p}_{a,d}\big(\phi_{a,d,\delta}\big) \le \mathrm{e}^\kappa + \sum_{n=1}^\infty a_n\cdot\frac{\delta_n}{2^n}\cdot \max_{x\in K}\max_{0\le k\le d_n}\big|\phi_n^{(k)}(x)\big| \le 1+\mathrm{e}^\kappa.$$

On the other hand,

$$P(\phi_{a,d,\delta}) = \phi_0(0)^2 + \sum_{n=1}^\infty \frac{\delta_n}{2^n}\cdot\phi_n(n)\cdot\phi_0^{(n)}(0) = \frac{1}{a_0^2} + \sum_{n=1}^\infty\frac{\delta_n}{2^n}\cdot\frac{\kappa^n}{a_n}.$$

Now consider $\delta_N = (1,1,\dots,1,0)$; then

$$P(\phi_{a,d,\delta}) = \frac{1}{a_0^2} + \sum_{n=1}^N \frac{\kappa^n}{2^n\cdot a_n} \ge$$

In the proof we have employed the following lemmas.

Exercise 5.4 Let M be a topological space, $p \in M$ and $\mathcal{B}$ be a basis of neighbourhoods of p. Prove the following:

i.) The set
$$J := \{(U, q) \colon U \in \mathcal{B}, q \in U\}$$
may be endowed with a direction by setting
$$(U, q) \preccurlyeq (U', q') \qquad \Longleftrightarrow \qquad U \supseteq U'.$$

ii.) The net
$$\phi \colon J \longrightarrow M, \quad \phi(U, q) := q$$
converges to p.

Exercise 5.5 Let $K \subseteq \mathbb{R}$ be a compact set, $a > 0$ and $d \in \mathbb{N}_0$. Construct a smooth function $\phi \in \mathscr{C}^\infty_K(\mathbb{R})$ such that
$$\max_{x \in K} \max_{0 \leq k \leq d} \left|\phi^{(k)}(x)\right| = a.$$

Hint: What we are really looking for is a bump function supported within K and with uniformly bounded derivatives up to order d. Once you acquire such a function, fine-tune it.

Notably, this contradicts [5, Lemma 2.1.9]. Indeed, its proof contains a fatal flaw within the final step: In a bornological space such as $\mathscr{C}^\infty_c(\mathbb{R})$, all bornivorous absolutely convex sets are zero neighbourhoods, but the preimage $P^{-1}(B)$ of an absolutely convex set under a polynomial is typically no longer convex. This already fails in two dimensions.

Example 5.6 Consider the quadratic polynomial
$$Q \colon \mathbb{C}^2 \longrightarrow \mathbb{C}, \quad Q(z, w) := zw.$$
Then $Q(z, 0) = 0 = Q(0, z)$ for all $z \in \mathbb{C}$ and thus
$$\mathbb{C} \times \{0\} \subseteq Q^{-1}\big(\mathrm{B}_r(0)\big) \supseteq \{0\} \times \mathbb{C}$$
for all $r > 0$. But e.g. $(r, 1) \notin Q^{-1}(\mathrm{B}_r(0))$, as $Q(r, 1) = r$ for all $r > 0$. Hence, $Q^{-1}(\mathrm{B}_r(0))$ is not convex.

If the domain is first countable, then boundedness does nevertheless imply continuity. Note that in this situation, the concepts of sequential continuity and continuity coincide.

Lemma 5.7 *Let $Q \colon V \longrightarrow W$ be a bounded polynomial between a first countable space V and a locally convex space W. Then Q is continuous.*

PROOF: We prove that discontinuous polynomials are unbounded. Without loss of generality, we may assume that Q is k-homogeneous for some $k \in \mathbb{N}$. By [1, Prop. 1.11], whose proof is analogous to Proposition 2.16, continuity of Q is equivalent to its continuity at the origin. Hence, we may assume that Q is discontinuous at zero, which yields a zero sequence $(v_n)_n \in c_0(V)$ such that its image $(Q(v_n))_n \notin c_0(W)$. By Lemma 2.12, we thus find a seminorm $\mathrm{q} \in \mathrm{cs}(W)$, a subsequence $(w_n)_n$ of $(v_n)_n$ and $r > 0$ such that
$$\mathrm{q}\big(Q(w_n)\big) \geq r > 0 \qquad \text{for all } n \in \mathbb{N}_0.$$

It is instructive – though not necessary – to first prove the result for normed spaces V. Indeed, in this case, we may rescale to obtain

$$\mathrm{q}\Big(Q\big(\tfrac{w_n}{\|w_n\|}\big)\Big) \geq \frac{r}{\|w_n\|^k} \overset{n\to\infty}{\longrightarrow} \infty,$$

as $(w_n)_n \in c_0(V)$. But $(w_n/\|w\|_n)_n \in \ell^\infty(V)$, which means that P is unbounded. Returning to the general case, we invoke Proposition 2.19, *iii.)* to find an ascending system of seminorms $\{\mathrm{p_n}\}_{n\in\mathbb{N}}$ for V. As $(w_n)_n \in c_0(V)$, Lemma 2.12 yields a subsequence $(w_n^{(1)})_n$ of $(w_n)_n$ such that

$$\mathrm{p}_1\big(w_n^{(1)}\big) \leq 1 \qquad \text{for all } n \geq 1.$$

For the very same reason, this sequence has a further subsequence $(w_n^{(2)})_n$ such that

$$\mathrm{p}_2\big(w_n^{(1)}\big) \leq \frac{1}{2} \qquad \text{for all } n \geq 2.$$

Proceeding like this inductively yields subsequences $(w_n^{(2)})_n$ such that

$$\mathrm{p}_\ell\big(w_n^{(\ell)}\big) \leq \frac{1}{\ell} \qquad \text{for all } n \geq \ell$$

for all $\ell \in \mathbb{N}$. Consider now the diagonal sequence $(x_n)_n$ given by

$$x_n \coloneqq w_n^{(n)} \qquad \text{for all } n \in \mathbb{N}.$$

By construction, $(x_n)_n$ is a subsequence of $(w_n)_n$, which means

$$\mathrm{q}\big(Q(x_n)\big) \geq r \qquad \text{for all } n \in \mathbb{N}.$$

This implies

$$\mathrm{q}\big(Q(n \cdot x_n)\big) = n^k \cdot \mathrm{q}\big(Q(x_n)\big) \geq n^k \cdot r \overset{n\to\infty}{\longrightarrow} \infty.$$

Finally, we check the boundedness of $(n \cdot x_n)_n$. Indeed, if $\ell \in \mathbb{N}$ and $n \geq \ell$, then we may estimate

$$\mathrm{p}_\ell(n \cdot x_n) \leq n \cdot \mathrm{p}_n(x_n) = n \cdot \mathrm{p}_n\big(w_n^{(n)}\big) \leq n \cdot \frac{1}{n} = 1$$

as our defining system is ascending. This covers almost all indices, which means

$$\sup_{n\in\mathbb{N}} \mathrm{p}_\ell(n \cdot x_n) < \infty.$$

Variation of $\ell \in \mathbb{N}$ asserts the boundedness of $(n \cdot x_n)_n$, completing the proof. □

Going beyond first countable spaces, our proof technique runs into the problem there may be too many seminorms to control. In view of Theorem 5.3, we may ask for sequential continuity at most.

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